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Flexural Stiffness in Simple Bending

Rigidez de Peças em Flexão Simples

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Abstract

The determination of displacements of a structure at Service and Ultimate Limit States demands the use of constitutive materials laws with due consideration of cracking, reinforcement tension stiffening and also plastic segments. As the safe dimensioning imposes, at critical sections, a top moment coming from magnified loads, simultaneously with reduced strengths, it then becomes evident that, in elements in simple bending, the bending moment at first yielding, determined without any reduction in strengths, is greater than the dimensioning one. This means that, in absence of axial compression force, the bending moment – curvature relation, which is the base to determine flexural stiffness, is closer, in most part of the structure, to the one obtained with linear materials laws (State II). This paper confirms, by several numerical comparisons, that it is correct and easier to use Hooke's law not only for the steel but also for the concrete. In simpler cases, expressions for an equivalent flexural stiffness are given, which is constant throughout the structural element and causes the same displacement at the center of the span, taking the ratio between cracking and maximum bending moments as free variable and the reinforcement as parameter.

Keywords: Reinforced concrete, Simple bending, Flexural stiffness, Tension stiffening.

Resumo

Na determinação de deslocamentos de estruturas, em serviço e na situação de cálculo, devem ser consideradas as leis constitutivas dos materiais, a fissuração do concreto, o enrijecimento da armadura tracionada, bem como os segmentos plastificados. Tendo em vista que o dimensionamento impõe um momento "teto" nas seções críticas, decorrente das ações majoradas e, simultaneamente, das resistências de cálculo (minoradas), é evidente que, nas peças em flexão simples, esse momento de cálculo é inferior àquele do início de escoamento da armadura tracionada (sem a minoração das resistências). Com isto, na ausência de força axial de compressão o diagrama momento-curvatura, de onde decorre a rigidez à flexão, aproxima-se, na maior parte da estrutura, do obtido no Estádio II (linear). Assim sendo, comprova-se neste trabalho, através de várias comparações numéricas, que é correto e mais fácil usar a rigidez à flexão decorrente da adoção da lei de Hooke para o concreto e para o aço. Nos casos mais simples determina-se, ainda, a rigidez equivalente dessas peças, constante e que leva à mesma flecha na seção central, em função do quociente entre os momentos de fissuração e máximo, tendo a taxa geométrica da armadura como parâmetro.

Palavras-chave*:* Concreto armado, Flexão simples, Rigidez à flexão, Enrijecimento da armadura tracionada.

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1 Introduction

The present work determines the flexural stiffness of cracked and non-cracked elements, which can be used to obtain instantaneous displacements, at Service (maximum permissible displacements in use of the structure) and at Ultimate Limit States (plastic rotations in elastic-plastic analysis). The procedure that was taken is the simplest one as possible, because Hooke's law can also be applied to the concrete. In simpler cases, expressions are given for an equivalent flexural stiffness, which results from a combination of stiffness of State I (without cracking in some parts of the element) and State II (cracking present in the remaining parts). Together with the solution given in [10] for columns, that stiffness becomes known in most frequent cases of plane bending.

2 Bending stiffness of States I and II

The flexural stiffness of a structural element can be determined using its flexural stiffness of State I, in those segments without cracking, and of State II, in the remaining cracked segments. See appendix 1. In State I, it is given by:

$$(\mathrm{EI})_{\mathrm{I}} = \mathrm{E}_{\mathrm{ci}}\mathrm{I}_{\mathrm{I}} \text{ if } \mathrm{M} \leq \mathrm{M}_{\mathrm{cr}} \quad \mathbf{(1)}$$

Bearing in mind that, for usual cross-sections (rectangular and even circular), the moment-curvature segment in naked State II (tensile stresses ignored) is also very close to a straight line passing through the origin, it is possible to approximate the flexural stiffness $(EI)_{II}$ to the product of the moment of inertia of the cracked cross-section by the concrete secant modulus – given in the NBR 6118, item 8.2.5, as $E_{cs} = 0{,}85E_{ci}$ –, therefore assuming elastic materials. The reinforcement tension stiffening, due to bond with slip between steel bars and surrounding concrete in the interval of two successive cracks, can be considered with a third factor, derived in [4]. Once the bending moment of inertia of the cracked cross-section is determined, the flexural stiffness of State II results from the product of the three mentioned factors, that is:

$$(\mathrm{EI})_{\mathrm{II}} = \mathrm{E}_{\mathrm{cs}}\mathrm{I}_{\mathrm{II}}\frac{\varepsilon_{\mathrm{syk}}}{\varepsilon_{\mathrm{sym}}} \text{ if } \mathrm{M} > \mathrm{M}_{\mathrm{cr}} \quad \mathbf{(2)}$$

where the reinforcement tension stiffening factor is given by:

$$\frac{\varepsilon_{\mathrm{syk}}}{\varepsilon_{\mathrm{sym}}} = \frac{1}{1 - \dfrac{0{,}18}{\rho_{\mathrm{s,ef}}}\dfrac{\tau_{\mathrm{bm}}}{\mathrm{f}_{\mathrm{yk}}}} \geq 1 \quad \mathbf{(3)}$$

This factor generally ranges between $1{,}05$ and $1{,}15$, and the value $1{,}1$ can be adopted, with little error, in more approximate calculations. The reinforcement tension stiffening model, adopted in the determination of Equation (3), is given in the CEB-FIP MC-90, items 3.2 and 7.4.3.1.1. This factor is based, as is apparent, on the first yielding of the tension reinforcement. This is a simplification made to obtain one expression only, to be used either in Service Limit States (SLS) verifications (instantaneous or initial displacements), or in elastic-plastic analysis (for plastic rotation demand) in Ultimate Limit States (ULS). In this model, the setback of the tension chord strain in the cracked section, ε_s, to the average strain, ε_{sm}, that is, $(\varepsilon_s - \varepsilon_{sm})$, is constant after the cracking formation phase and before the reinforcement yielding. This is not the case of the model adopted in the EC-2, item A2.2, and referred to in the book of Ghali and Favre, 1986, where this difference decreases with ε_s. See also [8] and [9]. In more rigorous calculations in SLS, this factor can be changed into the greater value $\varepsilon_s / \varepsilon_{sm}$, in the elastic segments of the structural element, where $\varepsilon_s = \sigma_s / E_s$ corresponds to the considered bending moment. This is equivalent to obtain, for each section in State II, a secant stiffness. It suffices, then, to put in (3) σ_s in place of f_{yk}.

The expressions given in Appendix 1 cover the most frequent cases appearing in practice, and refer to instantaneous displacements. The concrete creep and shrinkage, as well as the temperature actions are not considered here. See, for example, the mentioned book of Ghali and Favre.

If greater precision is needed in $(EI)_{II}$, it is better to use a constitutive law (for example, the Grasser law, given in the MC-90, item 2.1.4.4.1), which includes the concrete elasticity modulus, E_{ci}, tangent at the origin of the curve $\sigma_c(\varepsilon_c)$, as given in the NBR 6118, item 8.2.8.

Sigrist gives, in [9], the following expression, deduced by him, for the bending stiffness without compression reinforcement, resulting from the ratio between the moment at first yielding and the correspondent curvature (in the following, the original notation, f_c and f_y, is changed to f_{cm} and f_{yk}, respectively, to be coherent with the deduction of Equation (6)):

$$EI_{cr} = E_s bd^3 \omega(1-\omega)(1-\frac{\omega}{2})\frac{f_{cm}}{f_{yk}}\frac{\varepsilon_{syk}}{\varepsilon_{sym}} \quad \mathbf{(4a)}$$

where the mechanical reinforcement ratio is equal to:

$$\omega = \frac{A_s}{bd}\frac{f_{yk}}{f_{cm}} \quad \mathbf{(5)}$$

and

f_{yk} is the characteristic yielding strength of the steel;

f_{cm} is the mean value of the concrete strength (according to the MC-90, $f_{cm} = f_{ck} + 8$, in MPa).

This expression admits uniform stresses over the concrete compressed area, which means that the rectangular stress block relative depth ($\xi = x/d$) is equal to the mechanical reinforcement ratio. It has, therefore, one great advantage: it makes the search for the neutral axis unnecessary, which is already known through $\omega = \xi$. See Fig. 2.
If the stress block depth is considered equal to $y = 0{,}8x$, then the following expression results:

$$EI_{cr} = E_s \frac{bd^3}{8} \omega(2-\omega)(4-5\omega) \frac{f_{cm}}{f_{yk}} \frac{\varepsilon_{syk}}{\varepsilon_{sym}} \quad \textbf{(4b)}$$

If this expression is multiplied by $0{,}8$, therefore changing in the denominator 8 by 10, it is possible to obtain a good approximation with (6), for the case of rectangular cross-section without compression reinforcement.
The flexural stiffness of State II, deduced in [4], using the same ratio as Sigrist did, but with a second degree parabola law, $\sigma_c(\varepsilon_c)$, is given next by Equations (6) to (9), this time including compression reinforcement. The mechanical ratios are given by (5), that is, $\omega_1 = (A_{s1} f_{yk})/(bdf_{cm})$ and $\omega_2 = (A_{s2} f_{yk})/(bdf_{cm})$. Besides, the following dimensionless terms are used: $\beta_1 = \omega_1 \varepsilon_{c0} / \varepsilon_{syk}$ and $\beta_2 = \omega_2 \varepsilon_{c0} / \varepsilon_{syk}$, where ε_{c0} is the strain corresponding to the peak stress, f_{cm}, of the adopted second degree parabola $\sigma_c(\varepsilon_c)$. In this work, it was assumed that $\varepsilon_{c0} = 2^o/_{oo}$. With these data, one has:

$$(EI)_{II} = E_s bd^3 G(\xi) \frac{f_{cm}}{f_{yk}} \frac{\varepsilon_{syk}}{\varepsilon_{sym}} \quad \textbf{(6)}$$

$$G(\xi) = \frac{\omega_2}{\beta_2} \xi^2 \left(1 - \frac{\omega_2}{3\beta_2} \frac{\xi}{1-\xi}\right)\left(1 - \xi + \frac{a}{d}\right) + \omega_1 (\xi - \delta'_1)(1 - \delta'_1) \quad \textbf{(7)}$$

with the neutral axis relative depth determined by the following third degree equation:

$$\left(1 + \frac{\omega_2}{3\beta_2}\right)\xi^3 + (\beta_1 + \beta_2 - 1)\xi^2 - [\beta_1(1 + \delta'_1) + 2\beta_2]\xi + \beta_1 \delta'_1 + \beta_2 = 0 \quad \textbf{(8)}$$

In (7), the relative distance between the concrete compression force and the tensile reinforcement is equal to:

$$1 - \xi + \frac{a}{d} = 1 - \xi + \frac{\xi}{4}\left[\frac{8 - (8 + 3\frac{\omega_2}{\beta_2})\xi}{3 - (3 + \frac{\omega_2}{\beta_2})\xi}\right] \quad \textbf{(9)}$$

It has been assumed that the compression reinforcement does not yield. This would only occur if $x \geq 0{,}5d$. As shown next, the Ultimate Limit State imposes, as a limit to the neutral axis, exactly the value $0{,}5d$. It then becomes evident that, when the tensile reinforcement first yields, it must be $x < 0{,}5d$, because there is here no reduction in strengths, as there is in the Ultimate Limit State.
In Equations (6) to (9), the mechanical ratio should vary from a minimum value, corresponding to the minimum geometrical ratio, to a maximum value, estimated here only for concrete strength $f_{ck} / f_{cm} = 20/28\, MPa$. Assuming, initially, no compression reinforcement, the maximum neutral axis depth in the Ultimate Limit State, allowed nowadays in the NBR 6118, is $0{,}5d$, as already mentioned. Then, the maximum depth of the concrete stress block is $y = 0{,}8x = 0{,}4d$. From the condition of equal chord forces, that is, $A_s f_{yd} = by(0{,}85 f_{cd})$, it results, after dividing both sides by $bd(0{,}85 f_{cd})$, $\omega_d = (A_s f_{yd})/(0{,}85 f_{cd} bd)$. Thus, if the tensile reinforcement yields, this mechanical ratio equals the relative depth of the concrete stress block, that is, $\omega_d = y/d$, whose maximum is, consequently, $0{,}4$. With compression reinforcement, still in ULS, the neutral axis depth is kept equal to $x = 0{,}5d$. Therefore, the resistant bending moment increases exclusively due to the part corresponding to the steel section formed by additional compression and tension reinforcements. In it, the compression reinforcement is also admitted in yielding, and this occurs, for steel CA-50, if its relative depth verifies the inequality $\delta'_1 \leq 0{,}5(1 - \varepsilon_{yd}/0{,}0035) = 0{,}20$, where $\varepsilon_{yd} = f_{yd}/E_s = (500/1{,}15)/210 \times 10^3 = 0{,}00207$. See Fig. 1. This value results imposing $\varepsilon_{c1} = 0{,}0035$ and $\varepsilon_{s1} \geq \varepsilon_{yd}$. Introducing the parameter $k = A_{s1}/A_{s2} = \omega_{1d}/\omega_{2d} = \omega_1/\omega_2$, variable from 0 (no compression reinforcement) to 1 (symmetrical reinforcement), the compression reinforcement ratio is $k\omega_{2d}$, precisely equal to the

tensile reinforcement ratio in the steel section. Thus, the complete section has a tensile reinforcement ratio equal to:

$$\omega_{2d} = 0{,}4 + k\omega_{2d}$$

Therefore

$$\omega_{2d} = \frac{0{,}4}{1-k}$$

To reach the mechanical ratio ω_2, as defined by (5), based on this design mechanical ratio, ω_{2d}, it is admitted $f_{cm} / f_{ck} = 28/20 = 1{,}4$, $\gamma_s = 1{,}15$, $\gamma_c = 1{,}4$. Thus, the mechanical ratios ω_2 and ω_{2d} relate themselves in the proportion $1:2$, for:

$$\omega_2 = \omega_{2d}\frac{\gamma_s \times 0{,}85}{\gamma_c \times 1{,}4} = \omega_{2d}\frac{1{,}15 \times 0{,}85}{1{,}4^2} \cong \frac{\omega_{2d}}{2}$$

Consequently

$$\omega_2 \cong \frac{0{,}2}{1-k} \quad \textbf{(10)}$$

The NBR 6118, item 17.3.5.2.4, establishes the maximum area of compression plus tensile reinforcements equal to of the cross-section area, that is:

$$A_{s1} + A_{s2} \leq 0{,}04bh$$

Assuming that $h \cong 1{,}1d$ and introducing the mechanical ratios, with $f_{yk} = 500\ MPa$ and $f_{cm} = 28\ MPa$, the tension chord reinforcement limit results:

$$\omega_2 \leq 0{,}04 \times 1{,}1\frac{f_{yk}}{f_{cm}}\frac{1}{1+k} \cong 0{,}8\frac{1}{1+k} \quad \textbf{(11)}$$

The condition (10) equals this limit when $k = 0{,}6$. Thus, if the parameter k varies from 0 to $0{,}6$, the ω_2 limit imposed by (10) increases from $0{,}2$ to $0{,}5$. Taking k in the range $0{,}6$ to 1, condition (11) prevails. Under this condition, the total geometrical ratio is kept constant and equal to 4%, causing ω_2 to decay from $0{,}5$ to $0{,}4$. These limits have been approximately considered in Figures 2 and 3. Note that, when $k > 0{,}6$, the neutral axis depth in the ULS cannot be arbitrarily chosen anymore, because it has a unique and descending value for each ascending k.

As Fig. 2 shows, the comparison of the four presented solutions reveals that the elastic solution, according to Equation (2), is practically coincident with that from the parabolic law, according to (6), except for very high reinforcement ratios ($\omega > 0{,}30$), but this fact has no major relevance in design. On the other hand, Equation (4b) results in flexural stiffnesses, on the average, approximately $21{,}6\%$ greater than those from (6), in the range $\omega = 0{,}025$ to $0{,}20$. One may expect still greater closeness between the elastic and parabolic solutions, in doubly reinforced cross-sections, in T cross-sections and, also, in slabs. Indeed, Fig. 3 shows the dimensionless flexural stiffness determined from the elastic solution, Equation (2). The results have been compared with (6), and also with values given in [4]. The differences are very small.

Once the flexural stiffnesses of States I and II are known, it becomes possible, in the structural elements, to separate the cracked from the uncracked segments, using the cracking moment. In so doing, the corresponding curvatures can be determined. In structural elements with constant cross-section and reinforcement, the curvature varies, in each type of segment, uncracked and cracked, in the same way as the bending moment, but observing that, in the section where the acting moment equals the cracking moment, $M(x) = M_{cr}$, the curvature jumps from $M_{cr}/(EI)_I$ to $M_{cr}/(EI)_{II}$. This represents an approximate transition between States I and II, because the tensile stresses existing in the cross-section, including those from the cohesive crack, are disregarded. These stresses have been considered only in the cracking moment, M_{cr}, Equations (A1) and (A2a), and in the tension stiffening factor, according to (3), as mentioned before.

If the cracking moment is small ($\mu = M_{cr}/M_a \leq 0{,}4$, approximately), compared with the maximum moment occurring in the critical section, then it suffices to use the flexural stiffness of State II, as Figures 4 and 5 show.

3 Equivalent Flexural Stiffness

As an alternative to simpler cases, it is possible to determine an equivalent flexural stiffnesses for beams and slabs reinforced in one direction, which results from a combination of cracked and uncracked stiffnesses. In simply supported beams, with constant cross-section and longitudinal reinforcement, the following dimensionless terms are defined:

$\mu = M_{cr}/M_a \leq 1$: ratio between the cracking moment and maximum moment occurring in the element; **(12)**

$\beta = (EI)_I/(EI)_{II}$: ratio between flexural stiffnesses of States I and II; **(13)**

$\xi_{cr} = \dfrac{x_{cr}}{l} \leq \dfrac{1}{2}$: relative distance from the support to the section where $M(x) = M_{cr}$; **(14)**

$\xi_{cr} = [1 - \sqrt{1 - \psi}\,]/2$: for uniformly distributed load; **(15a)**

$\xi_{cr} = \psi / 2$: for concentrated load at the center of the span. **(15b)**

Once the flexural inertia moments of States I and II are known, the equivalent flexural stiffness, constant throughout the element, which produces the same transversal displacement at the center of the span, is obtained using the following expressions (see Figures 4 and 5):

$$(EI)_{eq} = E_{ci} I_I \frac{1}{\beta - 3{,}2(\beta - 1)(4 - 3\xi_{cr})\xi_{cr}^3} \quad \textbf{(16)}$$

for uniformly distributed loads. And

$$(EI)_{eq} = E_{ci} I_I \frac{1}{\beta - 8(\beta - 1)\xi_{cr}^3} \quad \textbf{(17)}$$

for concentrated load at the center of the span.
Both these equations verify the following boundary conditions:

(a) if $M_{cr} = M_a$, then $\psi = 1$, $\xi_{cr} = 0{,}5$, $(EI)_{eq} = E_{ci} I_I$, element entirely in State I.

(b) if $M_{cr} = 0$, then $\psi = \xi_{cr} = 0$, $(EI)_{eq} = E_{cs} I_{II} \dfrac{\varepsilon_{syk}}{\varepsilon_{sym}}$, element entirely in State II.

If there is no cracking, then $M_{cr} > M_a$, $\psi > 1$ and $(EI)_{eq} = E_{ci} I_I$.
Fig. 5 shows that the flexural stiffness of the structural element, for concentrated load at the center of the span, is more sensitive to the cracking moment than in the case of uniformly distributed load, Fig. 4. Note that this flexural stiffness is a good approximation in regions next to the internal support (negative bending moments), even for distributed load, if the parabolic bending moment of this region is close to a linear distribution.
In the following, the Branson's expression for the equivalent flexural stiffness, given in [3], is taken as a base for comparison. His expression, empirically determined for service conditions, can be put into the form:

$$(EI)_{eq} = E_{ci} \psi^m I_I + (1 - \psi^m) E_{cs} I_{II} = E_{ci} I_I [\psi^m + (1 - \psi^m) \frac{E_{cs} I_{II}}{E_{ci} I_I}] \quad \textbf{(18)}$$

where the exponent m equals 3 for distributed and concentrated loads. In the case of heavy concentrated load, $m = 4$ might be preferable (Branson, idem). Figures 6 and 7 show the comparison between Equations (16) and (17) with (18), for the two loading cases considered. It can be seen that, when $M_{cr} \geq 0{,}4 M_a$, the last one gives greater flexural stiffness, more pronounced for distributed load and low geometrical reinforcement ratios. Note that $\psi \geq 0{,}4$ means, for distributed load, cracking in $77{,}5\%$ of the span or *less*, because $\xi_{cr} \geq 0{,}113$, according to Equation (15a).
Table 1 shows the results of instantaneous displacements at the center of the span, obtained in the tests of Carbonari (see [5] and his other works), related to the concrete creep in sealed and non-sealed beams. The tests refer to linear creep, so that the concentrated load was chosen to result maximum compression stress in the concrete equal to $\approx 0{,}4 f_c$. As the concrete strength in the tests varies from $15\ MPa$ to $\approx 100\ MPa$, *for the same beam geometry*, consequently (and fortunately) the parameter $\psi = M_{cr}/M_a$ results in the range $0{,}4$ to $0{,}8$. With exception of one test only (that of number 4, with $f_c = 97\ MPa$), all other tests correspond to initial stresses in the reinforcement approximately between $120\ MPa$ and $320\ MPa$, and this would stand for service conditions in design. As can be seen in Table 1, the Branson's Equation, with $m = 3$, gives better results than those from Equation (17), here presented, although the correspondent deviation of 14%

in the mean value of the fraction a_1/a_0 is below the expected value ($\leq 20\%$). The reason for the small discrepancy lies in the factor chosen to represent the tension stiffening of the reinforcement, assumed in the present case equal to $\varepsilon_{syk}/\varepsilon_{sym}$, instead of the greater one, $\varepsilon_s/\varepsilon_{sm}$, as mentioned before. If this last one were chosen, then the mean value of the fraction a_1/a_0 would be $0{,}96$. ***Nevertheless, comparisons are still needed in the range of high stresses***, that represent the design situation in the ULS, for loads magnified by the partial safety factor $\gamma_f = 1{,}4$ or $1{,}5$ (and beyond these values), especially in the case of distributed loads, and with the parameter μ in the range $0{,}4$ to $\approx 0{,}95$.

When calculating displacements, the mean value of the concrete tension strength should be used. Table 2 shows an example of the influence of the axial tension strength in the equivalent stiffness, for a simply supported beam with uniformly distributed load. As a base for comparison, it is assumed the stiffness resulting from the mean value f_{ctm}, Equation (A2b). Although the cracking moment can vary in the proportion $1:2$, the stiffness varies in much smaller proportion. Indeed, taking f_{ctm}, and, for example, $\mu = M_{cr}/M_a = 0{,}6$, the error is small if, in the structure, the 5% fractile value occurs ($-4{,}2\%$ for the least geometrical ratio). On the other hand, if the 95% fractile value occurs, the error is greater ($13{,}1\%$, idem), but on the safe side, because the structure is stiffer than considered in design.

4 Closure

Concluding the present paper, the following aspects are stressed:

(a) The assumption of Hooke's law, for the concrete as well, leads to flexural stiffness practically coincident with those resulting from a second degree parabola law, except for very high mechanical ratios, an occurrence without major relevance in design.

(b) The given equations for the flexural stiffness include the reinforcement tension stiffening, and can be used either in an elastic-plastic analysis in ULS, or in SLS, in the calculation of instantaneous displacements. The equivalent flexural stiffness of columns, with rectangular cross-sections, is given in [10].

(c) The tension stiffening factor has been chosen, conservatively and as a simplification, based on the first yielding of the reinforcement. This results in a unique expression, which can be used either in SLS or in ULS.

(d) In simply supported beams, with uniformly distributed load, Fig. 4, or concentrated load at the center of the span, Fig. 5, expressions are given for the equivalent flexural stiffness, that consider both the cracked and uncracked segments.

(e) For calculation of design displacements, it is recommended to assume the mean value of the concrete tension strength, because the error in the stiffness is either small (on the unsafe side), or on the safe side.

(f) More comparisons with experimental results are needed to confirm the flexural stiffness, including prestressed concrete beams, with loads greater than those used in service conditions, especially for distributed loads and the factor $\mu > 0{,}4$. This would bring more precision in elastic-plastic analysis, where the demand of plastic rotation can be compared better and safer to the plastic rotation capacity.

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6 Appendix 1: Expressions for the flexural stiffness in States I and II

In the following, the most common expressions are given for the flexural stiffness in States I and II. In State I, one has

$$(EI)_I = E_{ci} I_I \quad \text{if } M \le M_{cr} = W_2 f_{ct,fl} \quad \textbf{(A1)}$$

where

$E_{ci} = 5600\sqrt{f_{ck}}$, in MPa , is the concrete elasticity modulus, tangent at the origin of its law $\sigma_c(\varepsilon_c)$, as given in the NBR 6118, item 8.2.8; f_{ck} is the concrete characteristic strength; I_I is the moment of inertia of the concrete cross-section, considering or not the reinforcement; W_2 is the section modulus, related to the extreme fiber in tension; and $f_{ct,fl}$ is the bending cracking strength of the concrete, related to the direct (axial) cracking strength f_{ct} (mean value). (See the comments and results given in Table 2). The mean value of the flexural tensile strength, originated by tension stresses acting in the section, including those of the cohesive cracking, is given by the following equation, according to the MC-90, item 2.1.3.3:

$$f_{ct,fl} = f_{ct}\frac{1+1{,}5(\frac{h}{100})^{0,7}}{1{,}5(\frac{h}{100})^{0,7}} \; , \; h \text{ in } mm \quad \textbf{(A2a)}$$

assuming the mean value of the tensile strength, given by:

$$f_{ctm} = 0{,}30 f_{ck}^{2/3} \; , \text{ in } MPa \quad \textbf{(A2b)}$$

As Equation (A2a) shows, this strength depends on the depth h of the structural element. Those elements with depth greater than $1500mm$, can already be considered fragile in tension, because, under such condition, both strengths, $f_{ct,fl}$ and f_{ct} , are practically coincident (both differ less than 10%). Equation (A2a) has been compared in [4] with two solutions: the original one from Sigrist [9], and the other derived from it, for the determination of the flexural tension strength using the cohesive crack model.

The reinforcement tension stiffening factor, derived in [4], is given by the following equation:

$$\frac{\varepsilon_{syk}}{\varepsilon_{sym}} = \frac{1}{1-\frac{0{,}18}{\rho_{s,ef}}\frac{\tau_{bm}}{f_{yk}}} \ge 1 \quad \textbf{(A3)}$$

where

$\tau_{bm} = 0{,}675 f_{ck}^{2/3}$ for short term load, and

$\tau_{bm} = 0{,}425 f_{ck}^{2/3}$ for permanent or repeated load, with f_{ck} in MPa ; $\rho_{s,ef} = \dfrac{A_s}{bh_{ef}}$ (beam) and $\rho_{s,ef} = \dfrac{a_s}{h_{ef}}$ (slab) are effective reinforcement ratios of the tension chord, with $h_{ef} = 2{,}5(h-d) \le h - \dfrac{x}{3}$ for beams with rectangular cross-sections and

$h_{ef} = 2{,}5(c + 0{,}5\phi_l) \le h - \dfrac{x}{3}$ for slabs, where a_s = reinforcement area per unit length, c = concrete cover and ϕ_l = diameter of the longitudinal steel bar.

The determination of the moment of inertia of State II is done for a T cross-section, Fig. 1, doubly reinforced, submitted to a bending moment greater than the cracking one. The following assumptions are made: (a) there is no axial force; (b) the neutral axis lies in the web, $x \ge h_{fl}$; (c) concrete in tension is neglected; (d) both materials follow Hooke's law; (e) the reinforcement compressed area is not deducted from the compressed area; (f) all variable and geometrical terms are positive. The dimensionless terms used in the equations are: $\alpha_s = E_s / E_c$ (ratio between elasticity moduli); $\rho_{s1} = A_{s1}/(b_w d)$ and $\rho_{s2} = A_{s2}/(b_w d)$ (geometrical ratios of compression and tensile reinforcement, respectively); and $\xi = x/d$ (neutral axis relative depth).

The neutral axis depth results from the equality of static moments of the compression and tension areas:

$$\frac{b_w x^2}{2} + (b_{fl} - b_w)h_{fl}(x - \frac{h_{fl}}{2}) + \alpha_s A_{s1}(x - d_1') = \alpha_s A_{s2}(d - x) \quad \textbf{(A4a)}$$

or

$$\xi^2 + 2[\alpha_s(\rho_{s1} + \rho_{s2}) + (\frac{b_{fl}}{b_w} - 1)\frac{h_{fl}}{d}]\xi - [2\alpha_s(\rho_{s1}\delta_1' + \rho_{s2}) + (\frac{b_{fl}}{b_w} - 1)(\frac{h_{fl}}{d})^2] = 0 \quad \textbf{(A4b)}$$

The moment of inertia referred to the neutral axis is:

$$I_{II} = \frac{b_w x^3}{3} + (b_{fl} - b_w)h_{fl}(x - \frac{h_{fl}}{2})^2 + (b_{fl} - b_w)\frac{h_{fl}^3}{12} + \alpha_s A_{s1}(x - d_1')^2 + \alpha_s A_{s2}(d - x)^2 \quad \textbf{(A5a)}$$

or

$$I_{II} = \frac{b_w d^3}{12}\{4\alpha_s[\rho_{s2}(1 - \xi)(3 - \xi) + \rho_{s1}(\xi - \delta_1')(\xi - 3\delta_1')] + (\frac{b_{fl}}{b_w} - 1)\frac{h_{fl}}{d}[(2\xi - \frac{h_{fl}}{d})(2\xi - 3\frac{h_{fl}}{d}) + (\frac{h_{fl}}{d})^2]\} \quad \textbf{(A5b)}$$

Both equations, (A4) and (A5), are only valid if $\xi = x/d \geq h_{fl}/d$. If this condition is not fulfilled, the calculation is repeated with $b_w = b_{fl}$, changing the reinforcement geometrical ratios as well.

Table 1 – Comparison of instantaneous deflections from theory and experiments. Tests of Carbonari, G., simply supported beams, span l =1800 mm, rectangular cross-section $b/h/d'$=100/150/20 or 30mm, A_s=2φ10=160mm², E^s=210GPa (assumed), concentrated load in the center of the span, $\tau bm = 0{,}675\ fc^{2/3}$. (The cylindrical strength fc is taken as the mean value of the strengths of 2 or 3 tested cylinders).

Test	f_c (MPa)	d' (mm)	G (kN)	a_0 (mm) Test	Theory $a_j = \dfrac{Gl^3}{48(EI)eq,j}$, $j = 1$ to 3						
					a_1 (mm) Eq. (17)	$\frac{a_1}{a_0}$	a_2 (mm) Eq. (18), m=3	$\frac{a_2}{a_0}$	a_3 (mm) Eq. (18), m=4	$\frac{a_3}{a_0}$	$\psi = \frac{M_{cr}}{M_a}$
1	45	20	13,26	4,57	4,11	0,90	4,14	0,91	4,49	0,98	0,399
2	45	20	11,70	3,50	3,56	1,02	3,45	0,99	3,82	1,09	0,452
3	15	20	5,60	1,74	2,08	1,20	2,01	1,16	2,16	1,24	0,494
4	97	30	16,50	4,90	4,45	0,91	4,22	0,86	5,10	1,04	0,504
5	35	20	8,80	2,01	2,73	1,36	2,53	1,26	2,84	1,41	0,517
6	27	20	6,64	1,90	2,07	1,09	1,85	0,97	2,09	1,10	0,587
7	38	30	7,78	2,41	2,54	1,05	2,18	0,90	2,55	1,06	0,594
8	27	30	5,75	1,43	1,94	1,36	1,61	1,13	1,87	1,31	0,651
9	20	20	4,65	1,01	1,40	1,39	1,20	1,19	1,34	1,33	0,703
10	24	30	4,77	1,35	1,51	1,12	1,20	0,89	1,37	1,01	0,731
11	23,5	30	4,32	1,06	1,23	1,16	0,97	0,92	1,08	1,02	0,797
				$(\Sigma a_i/a_0)/11=$		1,14		1,01		1,15	

Table 2 – Example of the influence of the direct (axial) cracking strength f_{ct} in the equivalent flexural strength. Other data according to Fig. 4.

ρs (%)	f_{ct}(MPa)	ψ=Mcr/Ma	$(EI)_{eq}/(E_{ci}I_I)$	Error (%)
	$f_{ctk,min}$=1,474	0,4	0,249	-4,2
0,5	f_{ctm}=2,210	0,6	0,260	0
	$f_{ctk,max}$=2,947	0,8	0,294	13,1
	$f_{ctk,min}$=1,474	0,4	0,383	-3,3
1	f_{ctm}=2,210	0,6	0,396	0
	$f_{ctk,max}$=2,947	0,8	0,438	10,6
	f_{ctkmin}=1,474	0,4	0,552	-2,3
2	f_{ctm}=2,210	0,6	0,565	0
	$f_{ctk,max}$=2,947	0,8	0,606	7,3

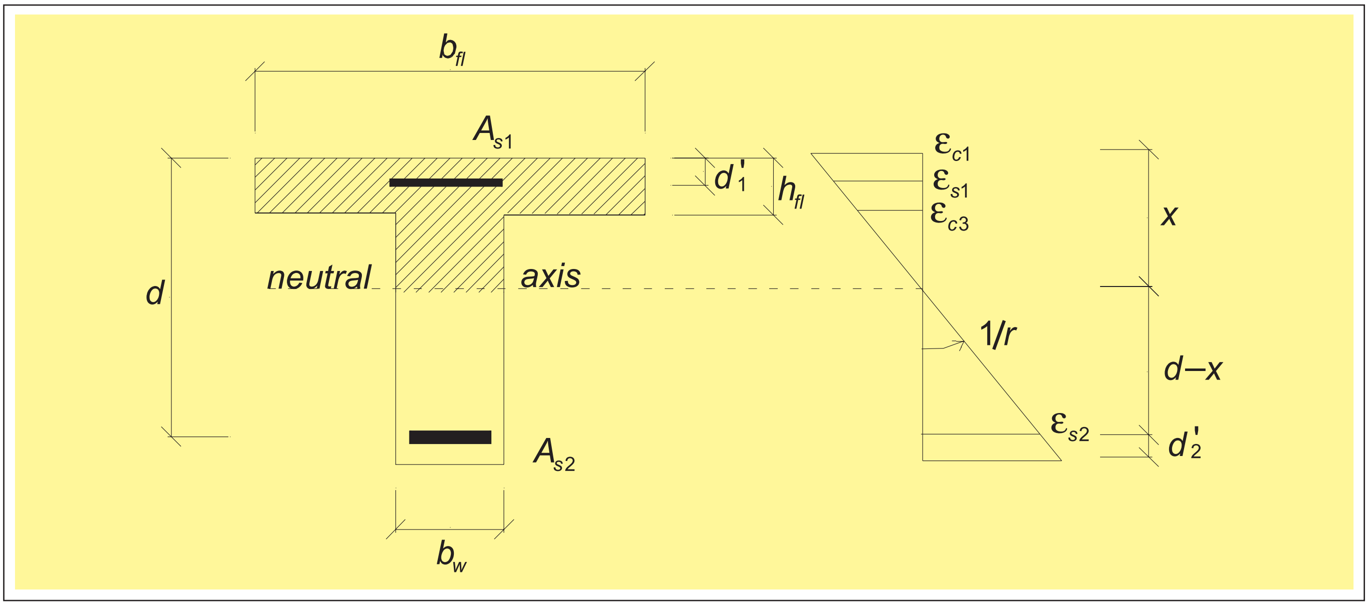

Fig. 1: T cross-section, doubly reinforced, State II.

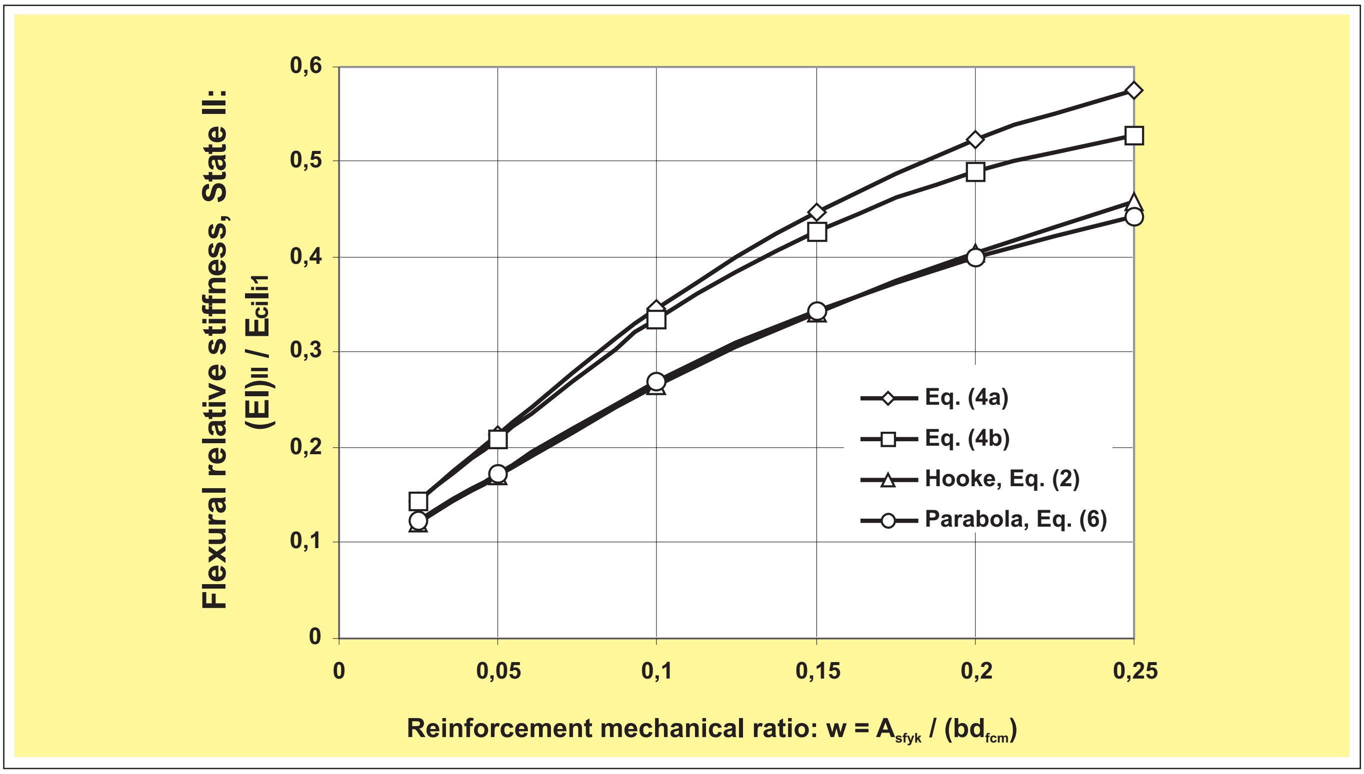

Fig. 2: Comparison of the solutions of the flexural relative stiffness of State II. Simple bending, rectangular cross-section, singly reinforced.

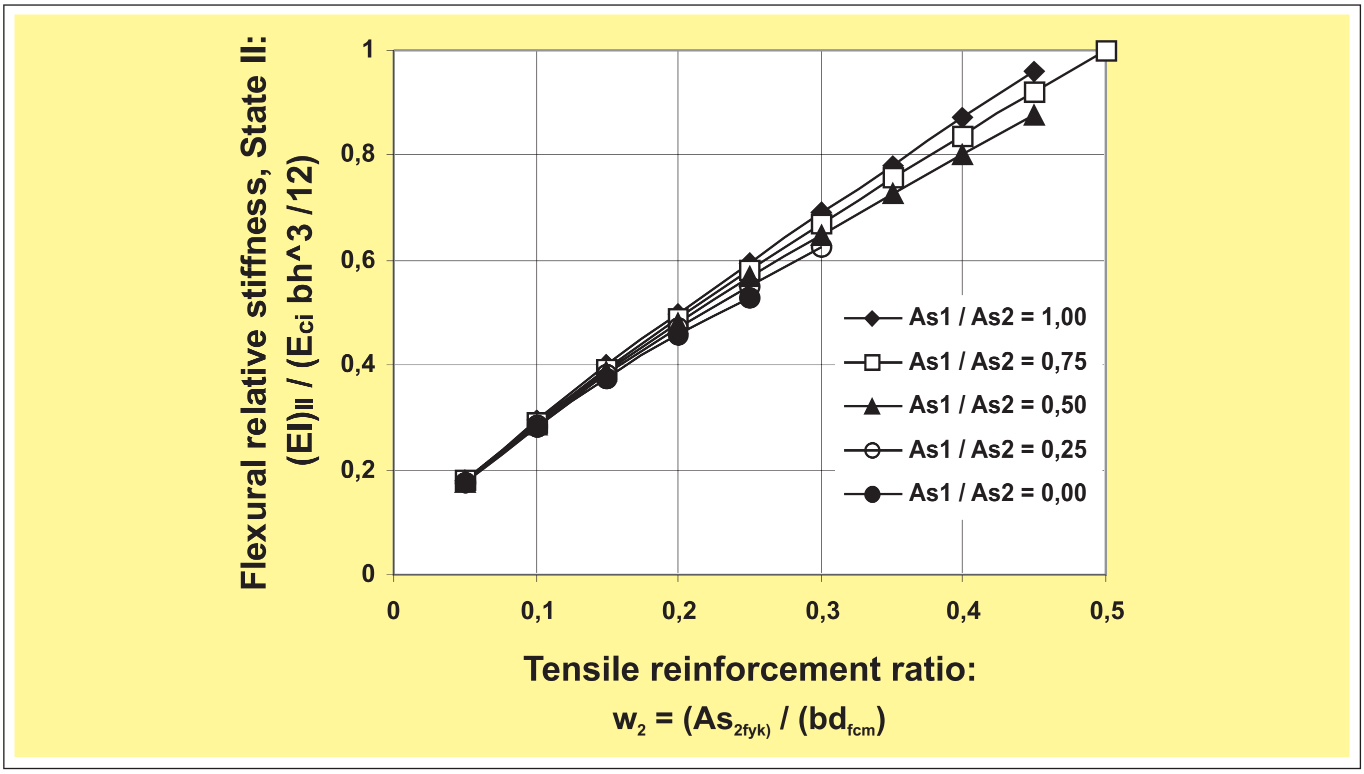

Fig. 3: Flexural relative stiffness of State II. Simple bending, rectangular cross-section, with and without compression reinforcement, $\delta_1' = d_1'/d = 0{,}1$ **and** $d/h = 0{,}9$ **.** $E_{ci} = 5600\sqrt{f_{ck}}$ **, in** MPa **.**

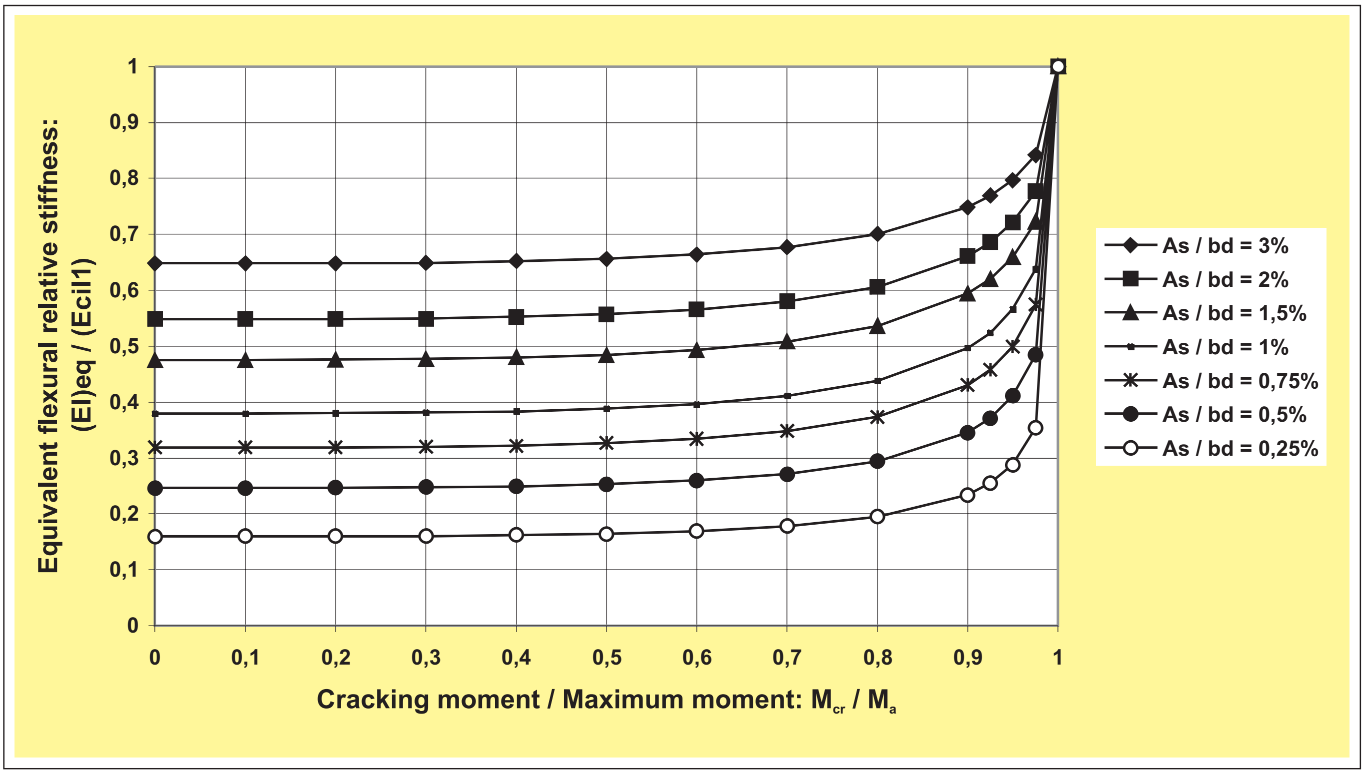

Fig. 4: Equivalent flexural relative stiffness as function of M_{cr}/M_a **, simply supported beam, with uniformly distributed load, Equation (16). Rectangular cross-section, singly reinforced,** $b/h/d = 200/500/450\ mm$ **,** $f_{ck} = 20\ MPa$ **.**

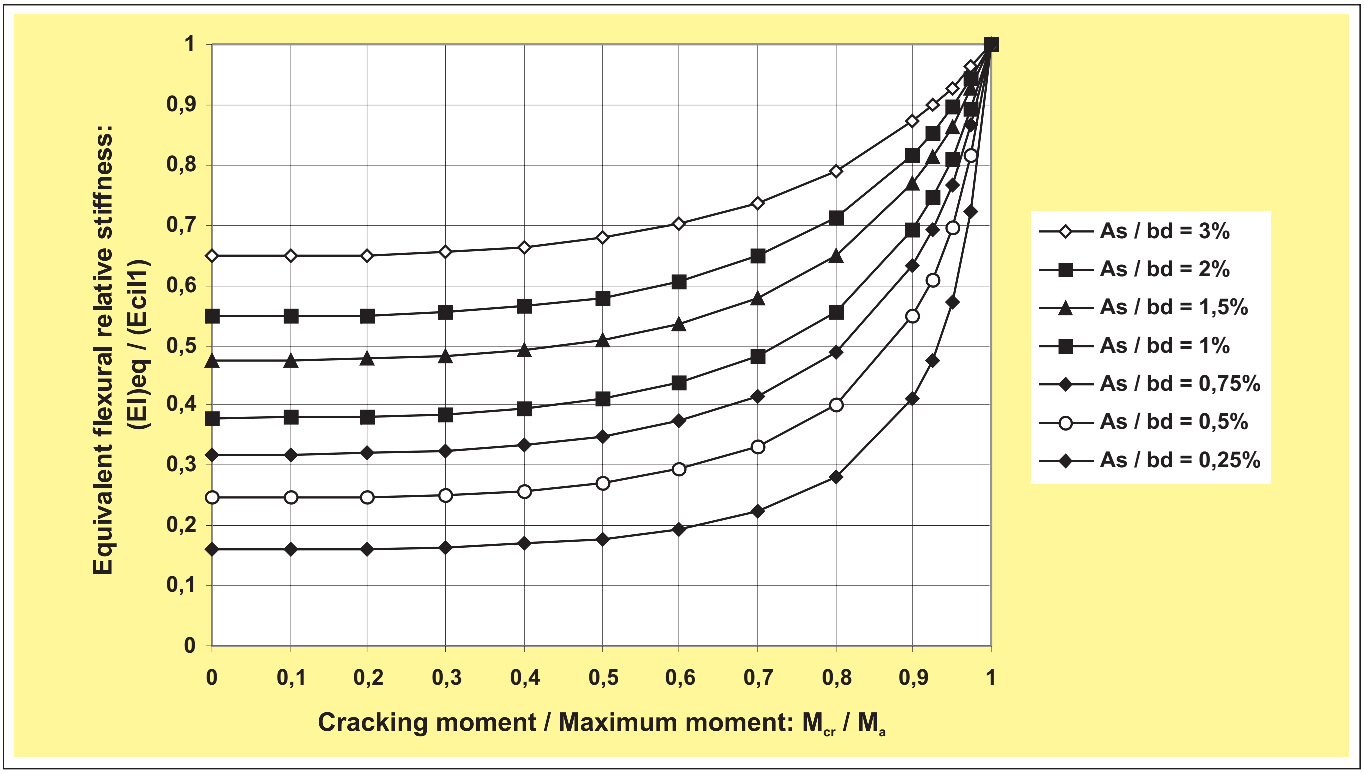

Fig. 5: Equivalent flexural relative stiffness as function of M_{cr} / M_a , simply supported beam, with concentrated load at the center of the span, Equation (17). Rectangular cross-section, singly reinforced, $b/h/d = 200/500/450\ mm$, $f_{ck} = 20\ MPa$.

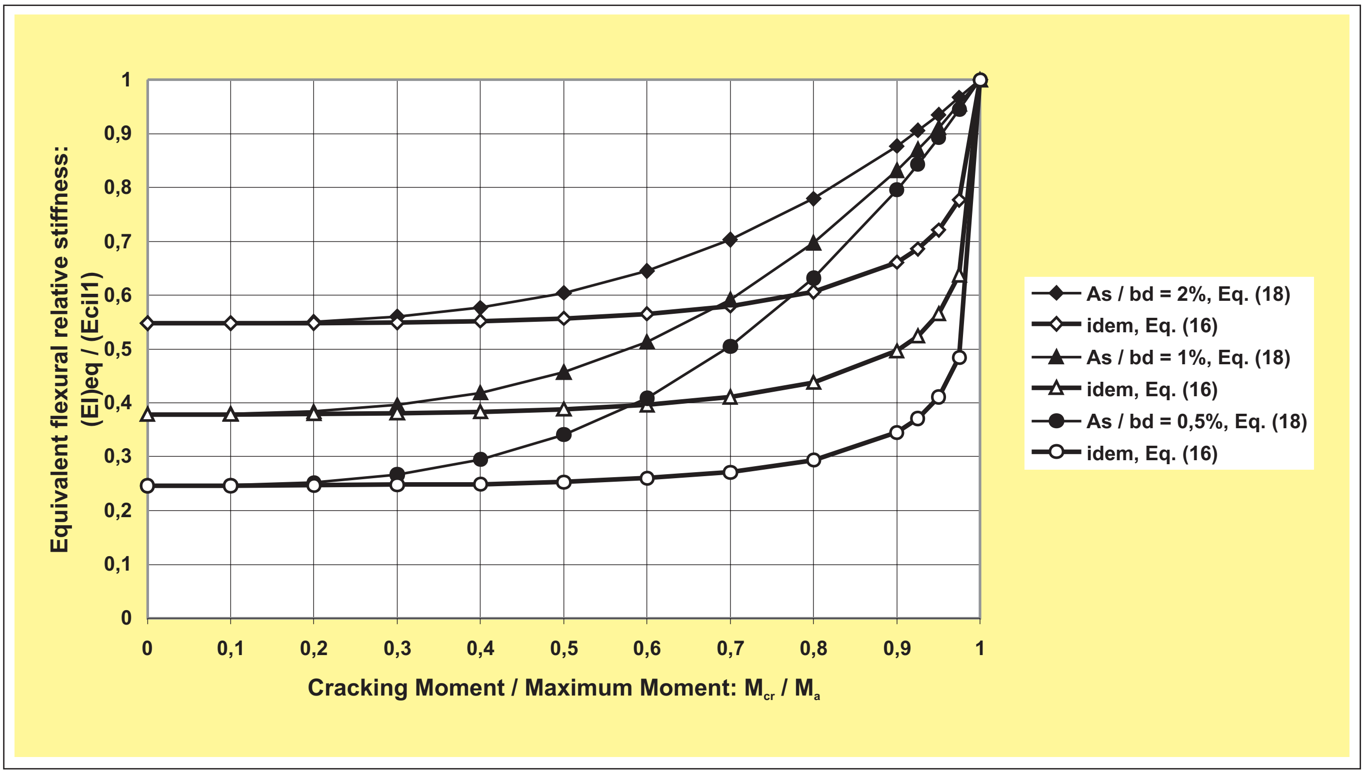

Fig. 6: Equivalent flexural relative stiffness as function of M_{cr} / M_a , Equations (16) e (18). Simply supported beam, with *distributed load*. Rectangular cross-section, singly reinforced, $b/h/d = 200/500/450\ mm$, $f_{ck} = 20\ MPa$, $f_{bm} = 0{,}425 f_{ck}^{2/3}$.

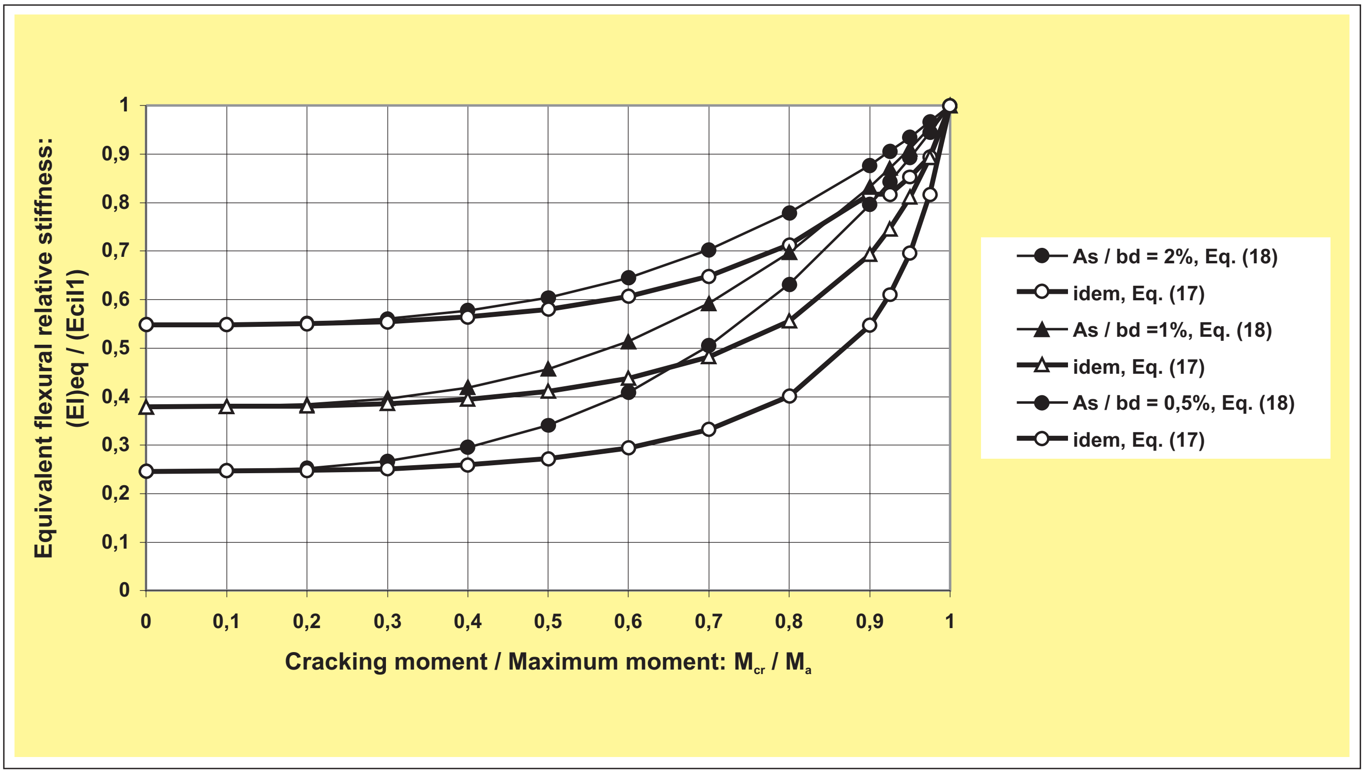

Fig. 7: Equivalent flexural relative stiffness as function of M_{cr} / M_a **, simply supported beam, with *concentrated load at the center of the span*, Equations (17) and (18). Rectangular cross-section, singly reinforced,** $b / h / d = 200 / 500 / 450\ mm$ **,** $f_{ck} = 20\ MPa$ **,** $\tau_{bm} = 0{,}425 f_{ck}^{2/3}$ **.**