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Design of Reinforced Concrete Two-dimensional Structural Elements: Membranes, Plates and Shells

Dimensionamento de Elementos de Superfície de Concreto Armado: Membranas, Placas e Cascas


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Abstract

Criteria and methodology for the design of concrete *membrane elements* with orthogonal reinforcement are presented. It is assumed that the internal forces (n_{Sdx}; n_{Sdy}, v_{Sd}) are determined from an elastic linear structural analysis. The theoretical fundamentals for reinforcement calculation and concrete crushing evaluation are reviewed. Also, the state of strain is evaluated, and it is proposed a procedure to estimate crack widths. Elastic linear structural analysis of *shell elements*, on the other hand, makes it possible to identify eight stress resultants: membrane (n_{Sdx}; n_{Sdy}, v_{Sd}), flexural (m_{Sdx}, m_{Sdy}, m_{Sdxy}) and transverse shear (v_x, v_y) forces. The three layer model proposed by CEB [1] provides a resistant mechanism for the design of elements with reinforcement consisting of a mesh of orthogonal bars. Procedures for reinforcement dimensioning and detailing are presented, considering the different layers capacities, the iterative process and the allowable simplifications. The computational procedure proposed by Lourenço & Figueiras [2] is implemented and critically reviewed.

Keywords: reinforced concrete, design, membranes, plates, shells.

Resumo

Neste trabalho são apresentados procedimentos para o dimensionamento de elementos de *membrana* com armaduras em malha ortogonal, dados os esforços solicitantes (n_{Sdx}; n_{Sdy}, v_{Sd}) provenientes de análise estrutural elástica-linear. É feita uma revisão dos fundamentos teóricos que orientam a determinação das armaduras necessárias e a verificação do concreto. O problema das deformações e da fissuração em elementos de membrana é abordado, sendo proposto um procedimento para a estimativa da abertura de fissuras. A análise estrutural elástica-linear de elementos de *casca*, por sua vez, permite a obtenção de oito resultantes de tensões: solicitações de membrana (n_{Sdx}; n_{Sdy}, v_{Sd}) e de placa (m_{Sdx}, m_{Sdy}, m_{Sdxy}), além de solicitações de cisalhamento transversal (v_x, v_y). O modelo de três camadas proposto pelo CEB [1] fornece um mecanismo resistente para o dimensionamento desses elementos com armadura em malha ortogonal. Procedimentos para orientar o dimensionamento e o detalhamento das armaduras baseados nesse modelo são apresentados, com o estudo das capacidades das diferentes camadas, dos diferentes braços de alavanca das forças internas, do processo iterativo e de simplificações possíveis.. O procedimento computacional para automatização do cálculo proposto por Lourenço & Figueiras [2] é implementado e avaliado.

Palavras-chave: concreto armado, dimensionamento, chapas, placas, cascas.ção Distribuída.

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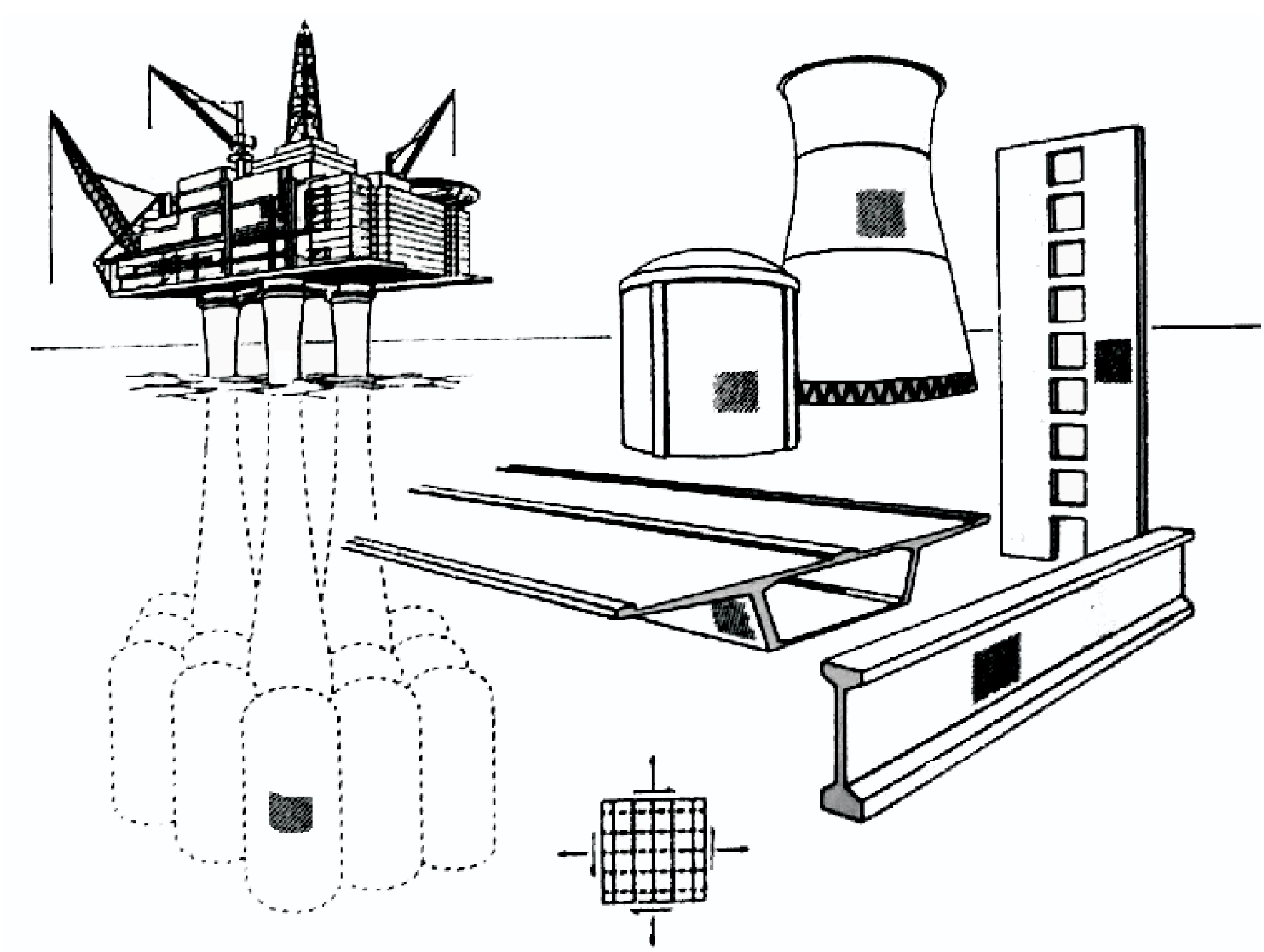
1 Introduction

Two-dimensional structural elements are basic components of many reinforced concrete structures such as bridges, nuclear plants and off-shore platforms (see figure [1]). They are represented by their mean plane and hold one dimension, generally called thickness, which, according to Corrêa & Ramalho [3], must be lesser than 0.25 times the length of the other two dimensions. The internal forces may be determined from a linear elastic analysis; however, the problem of verifying concrete crushing and calculating the required reinforcement still remains. In membrane, slab and shell elements, the reinforcement usually consists of a mesh of orthogonal rebars parallel to the *x*- and *y*-axes, directions not coincident to the principal stress directions. Cracks are then formed oblique to the reinforcement.
Until now, the solutions presented for the design of membrane elements subjected to in-plane actions (n_{Sdx}, n_{Sdy}, v_{Sd}) in recent structural codes as NBR-6118[4], ACI [5] and CEB [1] are either scarce and incomplete or not very clear. The load transfer mechanism in a cracked membrane element was studied by several researchers, such as Baumann (see [16]), Nielsen, Fialkow [6] and Gupta [7, 10]. In this paper, general procedures for the design of two-dimensional structural elements reinforced with two orthogonal sets of rebars are reviewed.
In a shell element two different actions may occur simultaneously: membrane actions (n_{Sdx}, n_{Sdy}, v_{Sd}) and plate actions (m_{Sdx}, m_{Sdy}, m_{Sdxy}), along with transverse shear (v_x, v_y). As for membrane elements, it is necessary to establish a model for the verification of shell and plate cracked elements with reinforcement consisting of an orthogonal net. Procedures for the design are still not well diffused. NBR-6118 [4] and Eurocode 2 [9] do not present any specific formulation for the design of shell elements; CEB [1], on the other hand, suggests the use of a three-layer model. In this model, the two outer layers are subjected to membrane action, while the inner layer establishes the shear transfer between the outer layers. CEB [1] states that an iterative procedure is required for the determination of both the thicknesses of each of

Figure 1 – Structures idealized as an assemblage of shell or membrane elements (Collins, 1986)



the three layers and the different lever arms between the internal forces, without giving further guidance however. The CEB itself [1] recognizes that *"...the exact determination of values... is complex and requires iterations, since it depends on the levels of the reinforcement and on the thickness of the concrete layers".*
Important contribution for the design of shell elements were given by Gupta [8], Lourenço & Figueiras [2], who proposed routines for the automatic design, and by the CEB [11, 12] in a bulletin on Ultimate Limit State design models. A more detailed study of the three-layer model is presented in this paper in order to provide guidance for its application in design procedures.

2 Brief description of the material

Concrete is a complex material, and a appropriate constitutive description of this material involves a large number of parameters. In this paper, however, it will be considered a rigid-plastic material, characterized by only one parameter, the effective compressive strength.
Verification of ULS – concrete. It should be verified that, under the ULS conditions, the maximum compressive force acting on an area of concrete does not exceed a limit value, corresponding to the resultant of the resistant stress given by the constitutive laws of the material and the safety factors. Nevertheless, simplifications of these constitutive laws are allowed. Alternatively to the parabola-rectangle stress-strain diagram, this work adopts the simplified uniform stress diagram over the full area of a zone under essentially uniaxial compression proposed by CEB [1]. The average stress for uncracked zones may be taken as:

$$f_{cd1} = 0{,}85.\left(1 - \frac{f_{ck}}{250}\right).f_{cd} \tag{2.1}$$

The concrete strength in the direction of the compressive stress is reduced after cracking due to the tensile stresses that are developed in the concrete between adjacent cracks and to the transmission of compressive stresses through previously formed cracks. Moreover, the concrete strips between the cracks are slender, and therefore less resistant to compression. The average concrete stress for cracked zones may be taken as:

$$f_{cd2} = 0{,}60.\left(1 - \frac{f_{ck}}{250}\right).f_{cd} \tag{2.2}$$

For concrete subjected to biaxial compression, the strength is increased to the value of $k.f_{cd1}$ due to concrete confinement. Also according to CEB [1]:

$$k = -\frac{1+3{,}80\alpha}{(1+\alpha)^2}, \text{ where } \alpha = \frac{\sigma_{2f}}{\sigma_{3f}} \text{ and } \sigma_{2f} \text{ and } \sigma_{3f} \text{ are principal stresses at failure} \tag{2.3}$$

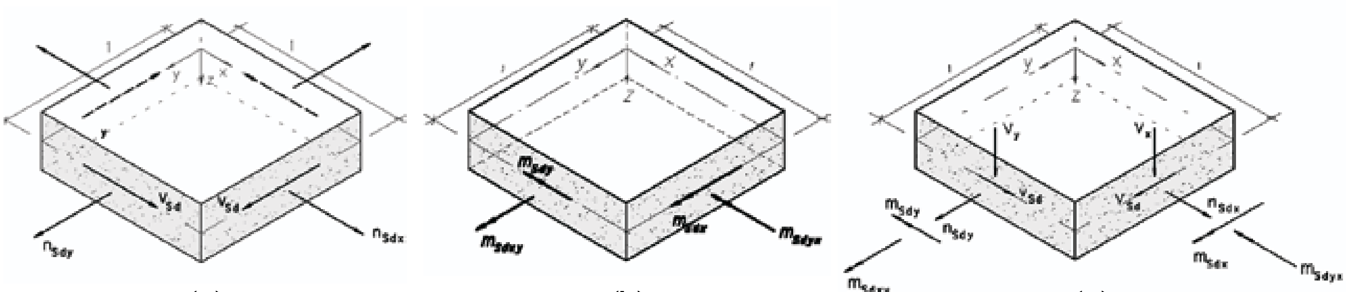
Verification of ULS – reinforcement. Reinforcement steel is considered a rigid-plastic material, with maximum stress equal to the yield stress. The contribution of the compressed steel is ignored, since it is small when compared with the resistance of the adjacent concrete.

3 Design of membrane elements

3.1 Design based on Baumann´s criteria

Consider a membrane element of thickness *h* with orthogonal reinforcement parallel to the *x*- and *y*-directions in its mean plane and subjected a set of applied forces (per unit length) n_{Sdx}, n_{Sdy}, v_{Sd} (figure [2a]), determined from a linear-elastic analysis. The principal directions, *1* and *2*, do not coincide with the reinforcement directions. Submitting the membrane to a significant stress state in which at least one of the principal stresses is a tensile one, parallel cracks will form, approximately linear in form and dis-

Figure 2 – Internal forces in a (a) membrane; (b) plate; (c) shell element (positive signs)



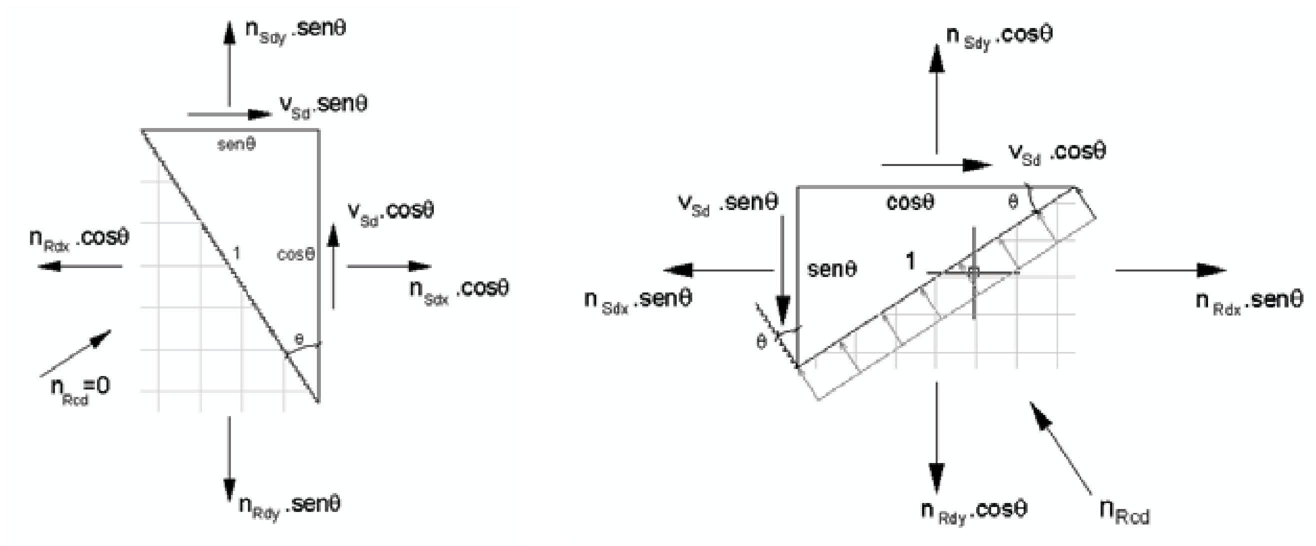
(a) (b) (c)

tributed in an evenly spaced pattern. The crack direction is conditioned by the existing reinforcements, and is generally oblique to them. Defining θ the angle between the crack direction and the y-direction, we usually have θ values different from zero.
It is assumed the following hypothesis: the forces acting on the membrane element are distributed approximately uniformly along the several rebars; the cracks are formed in a uniformly distributed and parallel pattern; the reinforcement resist axial loads only (dowel action is not considered); the tensile strength of concrete is negligible; perfect adhesion is considered between the reinforcement and concrete. Assigning by A_{sx} and A_{sy} the areas of reinforcement by unit length along y- and x-direction, respectively, we get the tensile forces resisted by the reinforcement, also by unit length:

$$n_{Rdx} = A_{sx}.\sigma_{sx}\,;\; n_{Rdy} = A_{sy}.\sigma_{sy}$$

where σ_{sx} and σ_{sy} are the stresses in the reinforcement in the x- and y-directions, respectively.
Equilibrium conditions of the internal forces in a membrane element adjacent to a crack, and with unit length along its direction, are then established. The horizontal and vertical dimensions of the element (catheters of the rectangle triangle), *senθ* and *cosθ*, are obtained directly from the trigonometric relationships as a function of θ. Since concrete-to-concrete friction along cracks is not considered, the concrete faces that are formed are free from any stress and, then, equilibrium condition gives (see figure [3a]):

Figure 3 – Equilibrium of internal forces in a membrane element (a) delimited by a plane parallel to the crack direction and (b) delimited by a plane orthogonal to the crack direction



(a) (b)

$$n_{Rdx} = n_{Sdx} + v_{Sd}.tg\theta \quad (3.1)$$

$$n_{Rdy} = n_{Sdy} + v_{Sd}.\cot g\theta \quad (3.2)$$

In planes orthogonal to the crack direction, uniformly distributed compressive stresses in concrete are developed between adjacent cracks. Let us consider, now, an element delimited by a plane orthogonal to the direction of compressive stresses (a plane orthogonal to the crack direction) as in figure 3b; the equilibrium condition in the direction of the compressive force on concrete gives:

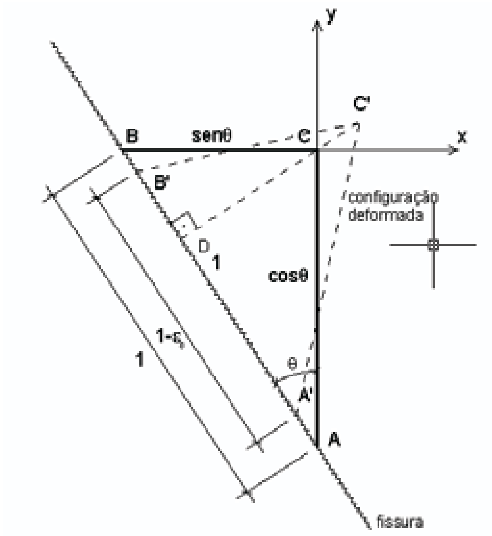
$$n_{Rcd} = v_{Sd}\left(tg\theta + \cot g\theta\right) \quad (3.3)$$

The solution of the problem is found considering, along with equilibrium conditions, compatibility conditions for the strains in both the reinforcement and concrete, as illustrated in figure 4:

$$DC'^2 = B'C'^2 - B'D^2 = A'C'^2 - A'D^2$$

$$(1+\varepsilon_{sx})^2 sen^2\theta - \left[(1-\varepsilon_c)sen^2\theta\right]^2 = (1+\varepsilon_{sy})^2\cos^2\theta - \left[(1-\varepsilon_c)\cos^2\theta\right]^2$$

Figure 4 – Strains in a membrane element with unit length parallel to the compressive stresses; geometrical relationships in the deformed element



$$B'C' = (1+\varepsilon_{sx})sen\theta$$
$$A'C' = (1+\varepsilon_{sy})\cos\theta$$
$$B'D = (1-\varepsilon_c)sen^2\theta$$
$$A'D = (1-\varepsilon_c)\cos^2\theta$$
$$A'B' = 1-\varepsilon_c$$

(a) (b)

where the terms ε^2, comparing to ε, are negligible, since they represent terms of superior order. Rearranging the terms, we get:

$$\frac{\varepsilon_{sy}}{\varepsilon_{sx}} = tg^2\theta\left[1+\frac{\varepsilon_c}{\varepsilon_{sx}}\left(1-\cot g^2\theta\right)\right],\ \text{onde}\ \varepsilon_{sx} = \frac{\sigma_{sx}}{E_s},\ \varepsilon_{sy} = \frac{\sigma_{sy}}{E_s},\ \varepsilon_c = \frac{\sigma_c}{E_c} = \left(\frac{d_c}{h}\right).\frac{1}{E_c}$$

$$\frac{\sigma_{sy}}{\sigma_{sx}} = tg^2\theta\left[1+\frac{d_c}{h.E_c}.\frac{A_{sx}.E_s}{n_{Rdx}}.\left(1-\cot g^2\theta\right)\right] \quad \textbf{(3.4)}$$

Optimum design

The optimum design is obtained when both sets of reinforcement are designed simultaneously to their maximum capacity, i.e., when the reinforcement in *x*- and *y*-directions develop the yield capacity.

$$\sigma_{sx} = \sigma_{sy} = f_{yd}$$

From (3.4), it is observed that the amount σ_{sx}/σ_{sy} is taken to unity when θ, the crack inclination relative to the *y*-direction, equals 45º. Thus, to consider a crack inclination of 45º leads to a design condition in which the reinforcement in both directions reaches the yield capacity and concrete is subjected to the minimum compression force, conditions that characterize the more economic solution. Equations (3.1) and (3.2) give:

$$n_{Rdx} = n_{Sdx} + v_{Sd}\,;\quad n_{Rdy} = n_{Sdy} + v_{Sd}$$

When the shear force is negative, cracks are formed so that the direction of v_{Sd} is opposite to that of n_{Rdx} and n_{Rdy} (see figure [5]). In this case, equilibrium conditions of the membrane element lead to:

$$n_{Rdx} = n_{Sdx} - v_{Sd}\,;\quad n_{Rdy} = n_{Sdy} - v_{Sd}$$

The two cases of loading previously exposed yields the following expressions:

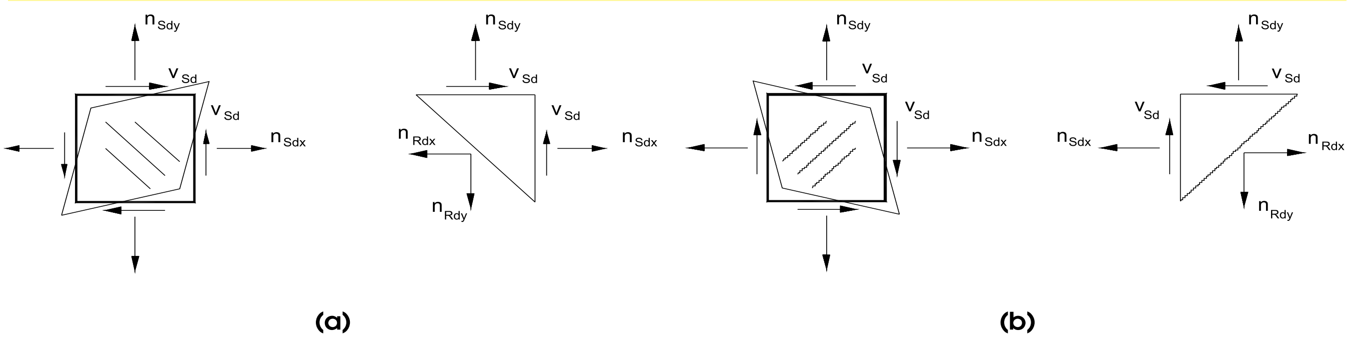
$$n_{Rdx} = n_{Sdx} + |v_{Sd}|\,;\quad n_{Rdy} = n_{Sdy} + |v_{Sd}| \quad \textbf{(3.5)}$$

From equations 3.1 and 3.2, we can distinguish the following dimensioning cases:

- ***Case I***

When $n_{Rdx} > 0$, $n_{Rdy} > 0$: $\qquad n_{Rdx} = n_{Sdx} + |v_{Sd}|\,;\quad n_{Rdy} = n_{Sdy} + |v_{Sd}|$

Figure 5 – Deformed configuration and crack pattern in a membrane element subjected to tensile and (a) positive or (b) negative shear forces



optimum area of reinforcement steel: $A_{sx} = \frac{n_{Rdx}}{f_{yd}}$; $A_{sy} = \frac{n_{Rdy}}{f_{yd}}$

concrete checking: $\frac{2v_{Sd}}{h} \leq 0{,}6\left(1 - \frac{f_{ck}}{250}\right)f_{cd}$

- ### Case II

When it is possible to omit the reinforcement in x-direction, the angle of the cracks relative to the y-axis is θ_o, different from the one calculated in the aforementioned optimum design, and can be determined from the action effects:

$$n_{Rdx} < 0,\ n_{Rdy} > 0: \qquad n_{Rdx} = 0;\ n_{Rdy} = n_{Sdy} - \frac{|v_{Sd}|^2}{n_{Sdx}},\ \text{with } tg\theta_o = -\frac{n_{Sdx}}{|v_{Sd}|}$$

area of reinforcement steel: $A_{sx} = 0$; $A_{sy} = \frac{n_{Rdy}}{f_{yd}}$

the compressive force in the concrete is: $n_{Rcd} = |v_{Sd}|(tg\theta_o + \cot g\theta_o) = |v_{Sd}|\left(-\frac{n_{Sdx}}{|v_{Sd}|} - \frac{|v_{Sd}|}{n_{Sdx}}\right) = -n_{Sdx} - \frac{|v_{Sd}|^2}{n_{Sdx}}$, and concrete

checking must be done by the equation: $\left(-n_{Sdx} - \frac{|v_{Sd}|^2}{n_{Sdx}}\right).\frac{1}{h} \leq 0{,}6\left(1 - \frac{f_{ck}}{250}\right)f_{cd}$

- ### Case III

Proceeding in a similar way as in the previous case, it is possible to find the expressions for the determination of the reinforcement in the x-direction when the reinforcement in the y-direction can be omitted. Equations (3.1) and (3.2) give:

$$n_{Rdx} > 0,\ n_{Rdy} < 0: \qquad n_{Rdy} = 0; \quad n_{Rdx} = n_{Sdx} - \frac{|v_{Sd}|^2}{n_{Sdy}},\ \text{with } \theta_o = arctg\left(-\frac{|v_{Sd}|}{n_{Sdy}}\right)$$

area of reinforcement steel: $A_{sx} = \frac{n_{Rdx}}{f_{yd}}$; $A_{sy} = 0$

concrete checking: $\left(-n_{Sdy} - \frac{|v_{Sd}|^2}{n_{Sdy}}\right).\frac{1}{h} \leq 0{,}6\left(1 - \frac{f_{ck}}{250}\right)f_{cd}$

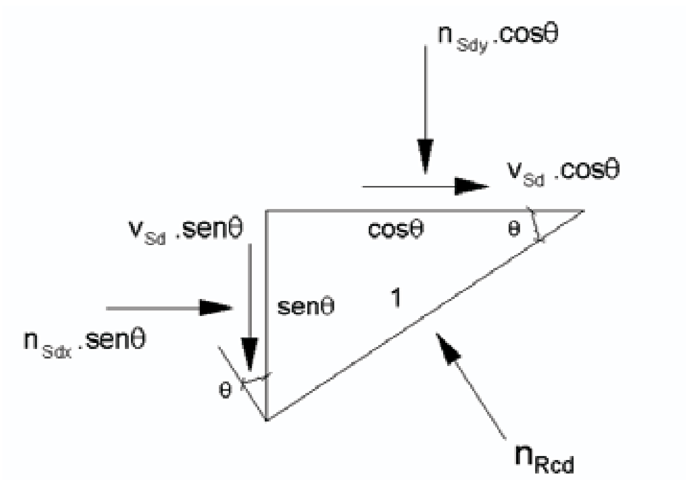
- ### Case IV

when the forces n_{Sdx} and n_{Sdy} are both compressive, its is possible to omit the reinforcement in the x- and y-directions, and we get from figure [6]:

$$n_{Rcd}.sen\theta - n_{Sdx}.sen\theta = v_{Sd}.\cos\theta \qquad (3.6)$$

$$n_{Rcd}.\cos\theta - n_{Sdy}.\cos\theta = v_{Sd}.sen\theta \qquad (3.7)$$

Figure 6 – Membrane element delimited by a vertical plane orthogonal to the compression forces without reinforcement in the x- and y-directions



and, (3.6) and (3.7) yields: $n_{Rcd} = \dfrac{n_{Sdx} + n_{Sdy}}{2} + \sqrt{\dfrac{(n_{Sdx} - n_{Sdy})^2}{4} + v_{Sd}{}^2}$

Therefore, when $n_{Rdx} < 0$, $n_{Rdy} < 0$: $n_{Rdx} = 0$; $n_{Rdy} = 0$

area of reinforcement steel: $A_{sx} = 0$; $A_{sy} = 0$

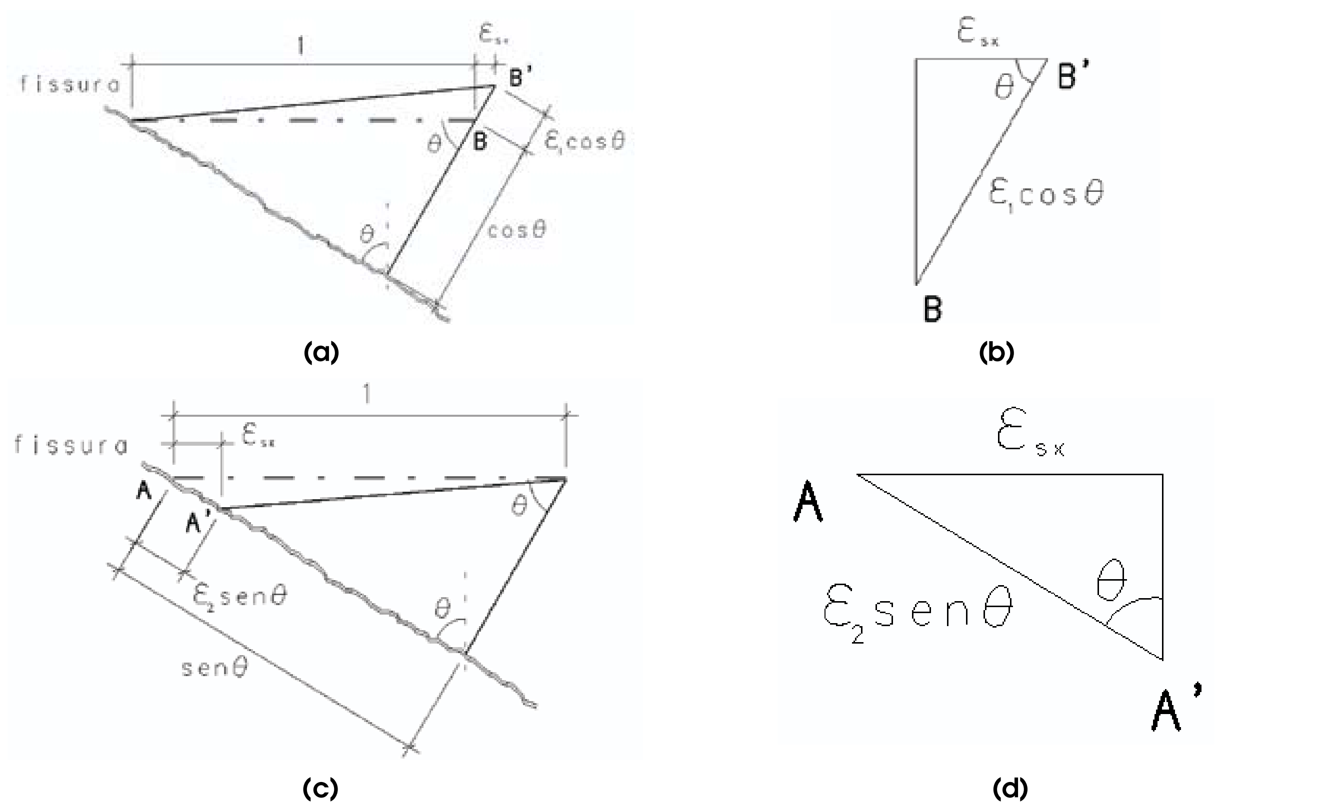
concrete checking: $\left(\dfrac{n_{Sdx} + n_{Sdy}}{2} + \sqrt{\dfrac{(n_{Sdx} - n_{Sdy})^2}{4} + v_{Sd}{}^2}\right)\dfrac{1}{h} \leq f_{cd1}$

3.2 State of strain

The study of the strains in a membrane element is important because it allows the determination of crack widths which, limited to acceptable values, assure that the structure will meet serviceability requirements. Thürlimann [20] and Gupta [7] gave contributions of great significance for the determination of the strains that may occur in a cracked membrane element reinforced with an orthogonal net of reinforcing steel. In this paper, only elastic strains due to the action of the applied loads will be considered (tension, compression and shear forces). Strains that may occur due to creep, shrinkage or expansion of the hardened concrete are not taken into account.
The study presented by Gupta in 1981 on the strains in a cracked membrane element takes into account compatibility conditions. In his formulation, it is assumed that the crack direction is perpendicular to the principal tensile strain direction in any loading stage. Consequently, the direction of ε_1 coincides with the direction of the mean crack strain, while the direction of the smallest principal strain, ε_2, is parallel to the crack direction. It is also admitted that the concrete is perfectly adhered to the reinforcement. In this way, strains experienced by the concrete in each loading stage must be associated to identical strains in the reinforcement so that compatibility will be respected.
Consider a membrane element; the length of its dimension in the *x*-direction is taken as unity. The strain in the reinforcement in the *x*-direction, ε_{sx}, is obtained from the sum of two components, one due to the principal tensile strain and the other one due to the compressive strain, according to figure [7].

$$\varepsilon_{sx} = \varepsilon_1 . \cos^2\theta + \varepsilon_2 . sen^2\theta \qquad \textbf{(3.8)}$$

Figure 7 – (a) strains in a membrane element subjected to the principal strain ε1; (b) point B displacements; (c) strains in a membrane element subjected to the principal strain ε2; (d) point A displacements



Consider now another membrane element; the length of its dimension in the *y*-direction is taken as unity. The strain in the reinforcement in the *y*-directions, ε_{sy}, is also obtained from the sum of two components, one due to the principal tensile strain and the other one due to the principal compressive strain, as shown in figure [8].

$$\varepsilon_{sy} = \varepsilon_1 . sen^2\theta + \varepsilon_2 . \cos^2\theta \qquad (3.9)$$

The strains in the direction of the reinforcements, *x* and *y*, are given by the equations (3.8) and (3.9), ε_1 and ε_2 being positive for tension and negative for compression. From these equations, it is possible to characterize the strain state of a membrane element in ultimate or service state conditions.

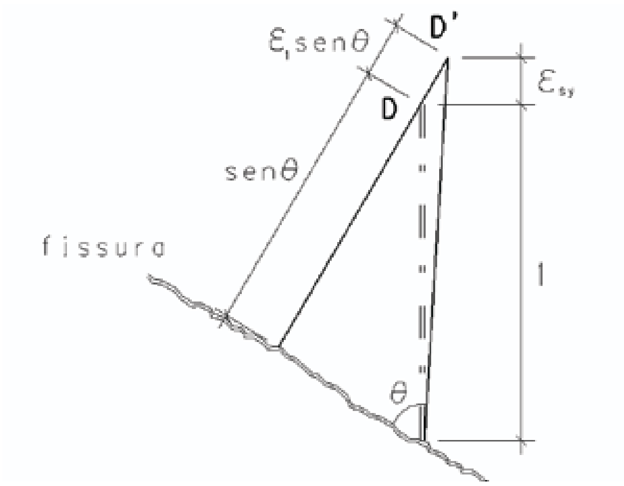
Limit state of deformation. When the ultimate limit state of the membrane element occurs by yielding of the reinforcement, two different failure modes may occur: development of the yield strength of the reinforcement in both *x*- and *y*-directions or development of the yield strength of just one set of reinforcement. When n_{Rdx} and n_{Rdy} are both positive, the deformations ε_{sx} and ε_{sy} should be positive. The optimum design occurs for the simultaneous yield of the two sets of reinforcement, i.e., $\varepsilon_{sx} \geq \varepsilon_y$ and $\varepsilon_{sy} \geq \varepsilon_y$, where ε_y is the yield strain of the reinforcing steel. For θ between 0° and 45°, $\varepsilon_{sx} \geq \varepsilon_{sy}$. If we set $\varepsilon_{sy} = \varepsilon_y$, the strain in the reinforcement in the *x*-direction remains greater than ε_y and the equation (3.9) gives:

$$\varepsilon_y = \varepsilon_1 . sen^2\theta + \varepsilon_2 . \cos^2\theta \;;\; \frac{\varepsilon_1}{\varepsilon_y} = \frac{1 + r.\cos^2\theta}{sen^2\theta} \text{, with } r = -\frac{\varepsilon_2}{\varepsilon_y} \qquad (3.10)$$

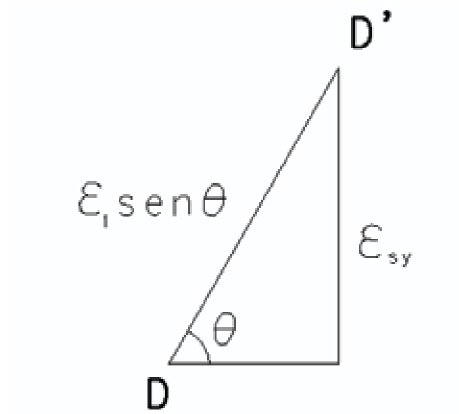
For θ between 45° and 90°, $\varepsilon_{sy} \geq \varepsilon_{sx}$. If we set $\varepsilon_{sx}=\varepsilon_y$, the reinforcement in the y-directions will also develop its yield capacity and equation (3.8) gives:

$$\varepsilon_y = \varepsilon_1 . \cos^2\theta + \varepsilon_2 . sen^2\theta \ ; \ \frac{\varepsilon_1}{\varepsilon_y} = \frac{1 + r.sen^2\theta}{\cos^2\theta}, \text{ with } r = -\frac{\varepsilon_2}{\varepsilon_y} \tag{3.11}$$

Figure 8 – (a) strains in a membrane element subjected to the principal strain ε1; (b) point D displacements; (c) strains in a membrane element subjected to the principal strain ε2; (d) point E displacements



(a)



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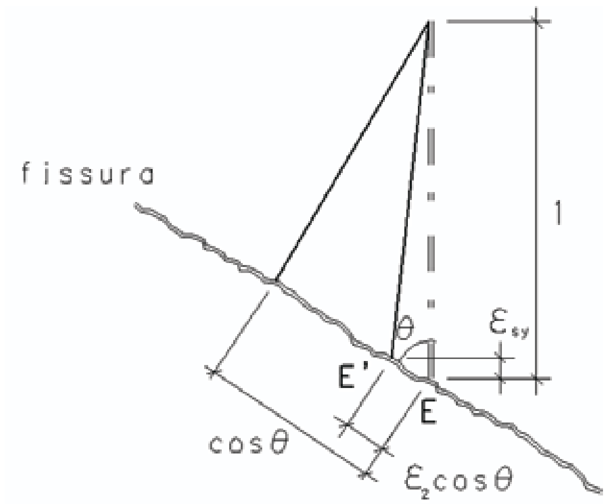
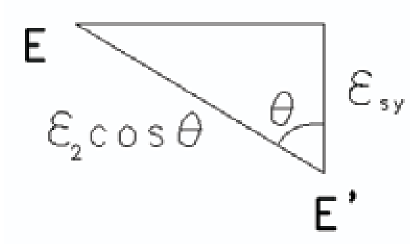


(c)



(d)

When n_{Rdy} is zero

$$tg\theta = -\frac{|v_{Sd}|}{n_{Sdy}} \tag{3.12}$$

Assuming yielding of the reinforcement in the x-directions with $\varepsilon_{sx}=\varepsilon_y$ and substituting in (3.8), we get:

$$\frac{\varepsilon_1}{\varepsilon_y} = \frac{1 + r.sen^2\theta}{\cos^2\theta}$$

which is the same expression given in (3.11), with the difference that it is now valid for any value of θ given by the expression (3.12). Similarly, when n_{Rdx} is zero,

$$tg\theta = -\frac{n_{Sdx}}{|v_{Sd}|} \tag{3.13}$$

Assuming yielding of the reinforcement in the y-direction with $\varepsilon_{sy}=\varepsilon_y$ and substituting in (3.9), we get:

$$\frac{\varepsilon_1}{\varepsilon_y} = \frac{1 + r.\cos^2\theta}{sen^2\theta}$$

which is the same expression given in (3.11), except for being valid for any value of θ given by equation (3.13).
Verification of serviceability limit states. In service conditions, the concrete should be cracked and the reinforcement in both directions with elastic behavior. The deformations in the reinforcement in x- and y-directions are given by:

$$\varepsilon_{sx} = \frac{n_{Rx}}{A_{sx}E_s} = \frac{n_{Sx} + v_S.tg\theta}{A_{sx}E_s};\ \varepsilon_{sy} = \frac{n_{Ry}}{A_{sy}E_s} = \frac{n_{Sy} + v_S.\cot g\theta}{A_{sy}E_s} \tag{3.14}$$

and the principal strain of the concrete in compression:

$$\varepsilon_2 = \frac{n_{Rc}}{hE_c} = \frac{v_S.(tg\theta + \cot g\theta)}{hE_c} \tag{3.15}$$

The deformations are given by the expression:

$$\varepsilon_1 + \varepsilon_2 = \varepsilon_{sx} + \varepsilon_{sy} \tag{3.16}$$

Equations 3.14 to 3.16, along with equation 3.8, yield:

$$\rho_y(1+\alpha.\rho_x)tg^4\theta + q_x.\rho_y tg^3\theta - q_y.\rho_x tg\theta - \rho_x(1+\alpha\rho_y) = 0 \tag{3.17}$$

where $\rho_x = \dfrac{A_{sx}}{h}$; $\rho_y = \dfrac{A_{sy}}{h}$; $q_x = \dfrac{n_{Sx}}{v_S}$; $q_y = \dfrac{n_{Sy}}{v_S}$; $\alpha = \dfrac{E_s}{E_c}$.

Since the value of θ is known, the steel and concrete strain can be determined by equations (3.14) and (3.15). It should be noted that the value of θ remains constant for different proportional values of n_{Sx}, n_{Sy} and v_S as long as the reinforcement remain with elastic behavior (Gupta [10]).

3.3 Verification of crack width

When the control of crack formation should be established through the limitation of crack widths, it is necessary to evaluate the crack widths from the strain state and, consequently, the tension state of the reinforcement of the membrane element under service loading. Most frequently, when dimensioning an element, the direction of the sets of reinforcement does not coincide with the principal tensile direction, and cracks will be formed in a direction oblique to the reinforcement. In this case, crack spacing and tensile strains of the cracked concrete element are different from those values calculated for elements subjected to uniaxial tension. Once the state of strain in the reinforcement is known, three proposals are presented for the application of expressions found in different codes for the determination of crack widths. The procedure for the verification of crack width is proposed below:

i) to calculate the reinforcement steel transverse section A_{sx} and A_{sy} according to the procedures described in 3.1;

ii) to determine the in-plane loading corresponding to the combination of actions considered for the verification in service load conditions: n_{Sx}, n_{Sy}, v_S;

iii) to determine the crack direction from equation (3.17);

$$\rho_y(1+\alpha.\rho_x)tg^4\theta + q_x.\rho_y tg^3\theta - q_y.\rho_x tg\theta - \rho_x(1+\alpha\rho_y) = 0$$

iv) with θ, to calculate the forces in the reinforcement and their respective strains. It should be emphasized that the reinforcement keeps in elastic behavior.

$$n_{Rx} = n_{Sx} + v_S tg\theta \ , \quad n_{Ry} = n_{Sy} + v_S \cot g\theta \ ; \ \varepsilon_{sx}, \ \varepsilon_{sy}$$

To determine, also, the major principal strain ε_1:

$$\varepsilon_2 = \frac{v_S.(tg\theta + \cot g\theta)}{hE_c}, \quad \varepsilon_1 + \varepsilon_2 = \varepsilon_{sx} + \varepsilon_{sy}$$

v) to determine the maximum crack spacing in the x- and y-directions and in the direction orthogonal to the crack. It may be calculated by the expressions (CEB, 1990) (Eurocode 2, 1999):

$$s_{\theta,\max} = \frac{1}{\dfrac{sen\theta}{s_{x,\max}} + \dfrac{\cos\theta}{s_{y,\max}}}, \ \text{with}\ s_{x,\max} = \frac{\phi_{sx}}{3{,}6\rho_{sx,ef}} \le \frac{\sigma_{sx}.\phi_{sx}}{3{,}6.f_{ct}} ; \quad s_{y,\max} = \frac{\phi_{sy}}{3{,}6\rho_{sy,ef}} \le \frac{\sigma_{sy}.\phi_{sy}}{3{,}6.f_{ct}}$$

Where $s_{x,max}$ and $s_{y,max}$ are the maximum cracking spacing in the x- and y- directions, respectively; s_{max} is the maximum space between cracks (mm); φ_s is the bar diameter; $\rho_{s,ef} = A_s/A_{c,ef}$; $A_{c,ef}$ is the effective tension area ($A_{c,ef}$ is, generally, the area of concrete surrounding the tension reinforcement of depth ($h_{c,ef}$) equal to 2,5 times the distance from the tension face of the section to the centroid of the reinforcement); $f_{ct,ef}$ is the tensile strength of concrete effective at the time when cracks may first be expected to occcur ($f_{ct,ef}=f_{ctm}$).

vi) to determine the crack width through one of the three following formulations. The first one is to calculate the crack width ignoring the elastic strains in the concrete between adjacent cracks. This formulation is conservative since it leads to crack widths that are larger than those that actually occur. The other two proposals consider the contribution of the concrete between cracks (tension-stiffening): to calculate the crack width using the expression given by the Eurocode 2 (1999) or the expression proposed by NBR-6118 (2003) for the determination of crack widths in linear elements in an adapted form.

a) CEB (1990) defines the characteristic crack width w_k as:

$$w_k = s_{\theta,\max}.\varepsilon_1$$

b) Eurocode 2 (1999) states that the design crack width may be obtained from the relation:

$$w_k = s_{\theta,\max}.(\varepsilon_{sm} - s_{cm}) \qquad (3.18)$$

Where $s_{\theta,max}$ is the maximum crack spacing, ε_{sm} is the mean strain in the reinforcement, taking into account the effects of the tension-stiffening, ε_{cm} is the mean strain in concrete between cracks. (ε_{sm}-ε_{cm}) may be calculated from the expression:

$$\varepsilon_{sm}-\varepsilon_{cm}=\varepsilon_1-\frac{0{,}4\dfrac{f_{ct,ef}}{\rho_{s,ef}}(1+\alpha_e\rho_{s,ef})}{E_s}\geq 0{,}6\frac{\sigma_s}{E_s} \quad (3.19)$$

where α_e is the ratio E_s/E_c. The reinforcement ratio in the principal direction of an element fitted with reinforcement bars running along two orthogonal directions x and y, with geometric ratios ρ_x and ρ_y, respectively, may be calculated from the following expression, according to CEB [11]:

$$\rho_{s,ef}=\rho_{sx,ef}\cos^2\theta+\rho_{sy,ef}sen^2\theta \quad (3.20)$$

c) According to NBR-6118 [4], the crack width in linear elements, determined for each part surrounding the tension reinforcement, shall be the smallest between the values obtained from the two expressions that follows:

$$w\leq\frac{\phi_i}{(20\eta_i-7{,}5)}\frac{\sigma_{si}}{E_{si}}\frac{3\sigma_{si}}{f_{ct,m}}\;;\;w\leq\frac{\phi_i}{(20\eta_i-7{,}5)}\frac{\sigma_{si}}{E_{si}}\left(\frac{4}{\rho_{ri}}+45\right)$$

Alternatively, the authors propose the analysis of the previous expressions as the product of the crack spacing and the principal tension strain:

$$w\leq\frac{\phi_i}{(20\eta_i-7{,}5)}\frac{\sigma_{si}}{E_{si}}\frac{3\sigma_{si}}{f_{ct,m}}=\left[\frac{\phi_i}{(20\eta_i-7{,}5)}\frac{3\sigma_{si}}{f_{ct,m}}\right].\varepsilon_1=s^*_{\max}.\varepsilon_1$$

$$w\leq\frac{\phi_i}{(20\eta_i-7{,}5)}\frac{\sigma_{si}}{E_{si}}\left(\frac{4}{\rho_{ri}}+45\right)=\left[\frac{\phi_i}{(20\eta_i-7{,}5)}\left(\frac{4}{\rho_{ri}}+45\right)\right].\varepsilon_1=s^*_{\max}.\varepsilon_1$$

and, doing so, the crack width can be obtained by the expression:

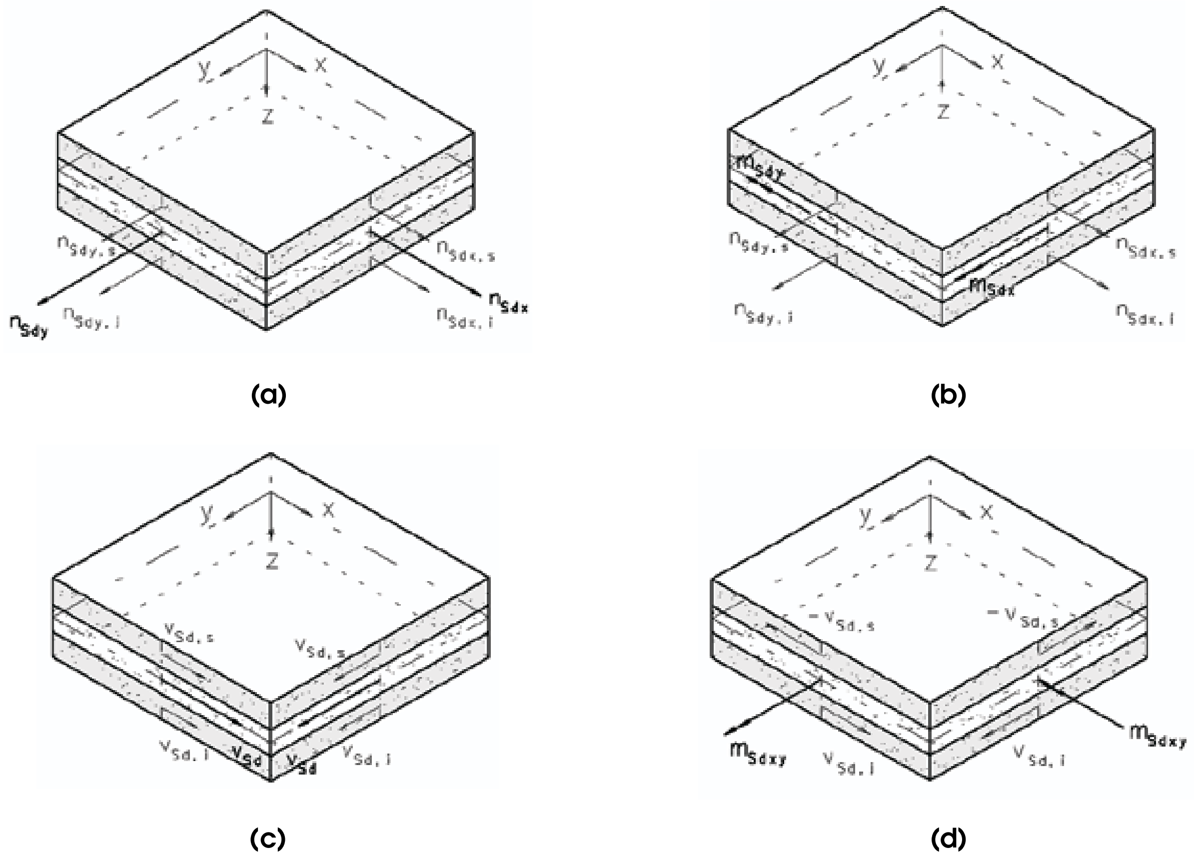
$$w_k=s^*_{\theta,\max}.\varepsilon_1\text{, with}$$

$$s^*_{x,\max}=\frac{\phi_{xi}}{(20\eta_{xi}-7{,}5)}\frac{3\sigma_{sxi}}{f_{ct,m}}\leq\frac{\phi_{xi}}{(20\eta_{xi}-7{,}5)}\left(\frac{4}{\rho_{rxi}}+45\right);$$

$$s^*_{y,\max}=\frac{\phi_{yi}}{(20\eta_{yi}-7{,}5)}\frac{3\sigma_{syi}}{f_{ct,m}}\leq\frac{\phi_{yi}}{(20\eta_{xi}-7{,}5)}\left(\frac{4}{\rho_{ryi}}+45\right);$$

$$s^*_{\theta,\max}=\frac{1}{\dfrac{sen\theta}{s^*_{x,\max}}+\dfrac{\cos\theta}{s^*_{y,\max}}}$$

Figure 9 – (a) normal actions and (b) bending moments in the shell element resolved into membrane actions in theouter layers; (c) shear forces and (d) torsional moments resolved into membrane actions in the outer layers



4 Design of shell elements

4.1 Definition of the resistant mechanism

Shell elements are subjected to combined membrane and slab forces components. Consider a shell element of infinitesimal dimensions *dx* and *dy*, parallel to the *x*-and *y*-directions taken as unity (figure 2c). In general, we can distinguish eight internal force components, which represent stress resultants acting on the element:

- 3 membrane components: n_{Sdx}, n_{Sdy}, v_{Sd};
- 3 slab components: m_{Sdx}, m_{Sdy}, $m_{Sdxy}=m_{Sdyx}$;
- 2 transverse shear forces: v_x, v_y.

These components are also illustrated in figure 2c (positive convention). The orthogonal *x*- and *y*-directions coincide with the plane of the shell, while the direction *z* is perpendicular to this plane.

According to the three-layer model proposed by CEB[1], a shell element may be modeled as three comprising layers bearing different functions: the two outer layers provide resistance to the normal effects of both the in-plane (n_{Sdx} and n_{Sdy}, see figure 9a) and bending (m_{Sdx} and m_{Sdy}, see figure 9b) loading and to the tangential effects of both the shear force (v_{Sd}, see figure 9c) and torsional moment (m_{Sdxy}, see figure 9d); the inner layer, on the other hand, must provide resistance to transverse shear forces v_x and v_y acting perpendicularly to the element's middle plane. It is assumed that each layer has uniform thickness: t_s for the top layer, t_i the bottom layer and t_c for the inner layer, so that:

$$t_s + t_c + t_i = h \tag{4.1}$$

The membrane forces per unit length parallel to orthogonal reinforcements in the upper layer (referred to by the index ',*s*') and lower layer (referred to by the index ',*i*') are calculated by the equations:

$$n_{Sdx,s} = n_{Sdx}\frac{z_x - y_{x,s}}{z_x} - \frac{m_{Sdx}}{z_x}\,;\quad n_{Sdx,i} = n_{Sdx}\frac{z_x - y_{x,i}}{z_x} + \frac{m_{Sdx}}{z_x} \tag{4.2}$$

$$n_{Sdy,s} = n_{Sdy}\frac{z_y - y_{y,s}}{z_y} - \frac{m_{Sdy}}{z_y}\,;\quad n_{Sdy,i} = n_{Sdy}\frac{z_y - y_{y,i}}{z_y} + \frac{m_{Sdy}}{z_y} \tag{4.3}$$

$$v_{Sd,s} = v_{Sd}\frac{z_{xy} - y_{xy,s}}{z_{xy}} - \frac{m_{Sdxy}}{z_{xy}}\,;\quad v_{Sd,i} = v_{Sd}\frac{z_{xy} - y_{xy,i}}{z_{xy}} + \frac{m_{Sdxy}}{z_{xy}} \tag{4.4}$$

where, according to the figure 10, z_x nd z_y are lever arms referring to the bending moments and the membrane's normal forces; z_{xy} is the lever arm referring to twisting moments and membrane shear forces; $y_{x,s}$, $y_{x,i}$, $y_{y,s}$, $y_{y,i}$ are distances between the mean plane of the element and the reinforcement's gravity centre in the two direction *x* and *y* for the absorption of the bending moments and the membrane's normal forces, so that we get $z_x = y_{x,s} + y_{x,i}$ and $z_y = y_{y,s} + y_{y,i}$; $y_{xy,s}$, $y_{xy,I}$ are distances between the medium plane of the layer and the reinforcement's gravity centre in the two *x*- and *y*-directions for the absorption of twisting moments so that we get $z_{xy} = y_{xy,s} + y_{xy,i}$. None of the lever arms should be greater than the distance between the reinforcement's gravity centers of the opposite sides. The transverse steel section required and the verification of the concrete in both upper and lower layers can then be carried out as for the design of membrane elements. However, the exact determination of the different thicknesses of the three layers must be done through an iterative procedure.

4.2 Dimensioning the Outer Layers

The outer layers should be designed according to the criterions exposed in subsection 3.1.

Figure 10 – Distances of the reinforcement's gravity centre in the x- and y-directions

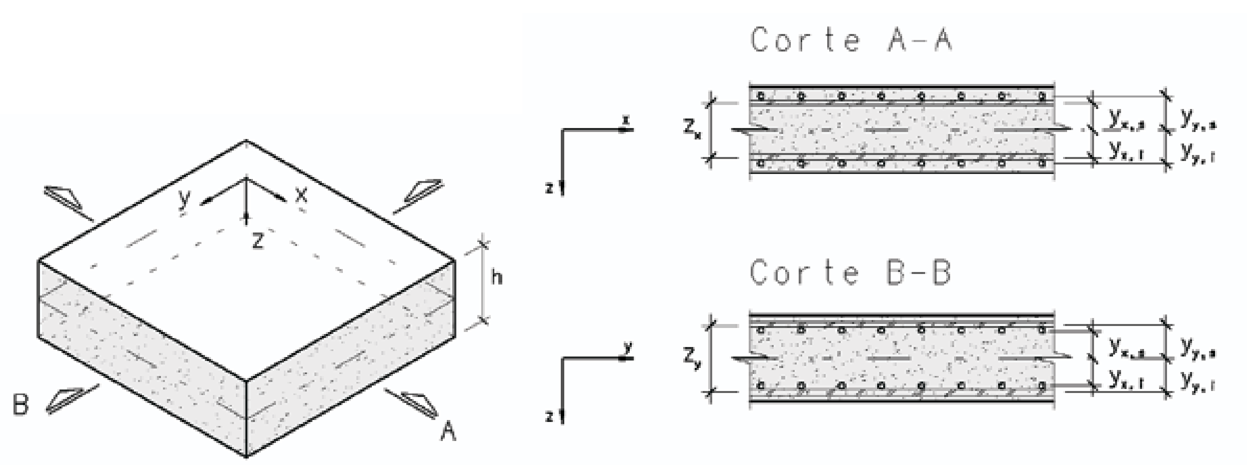
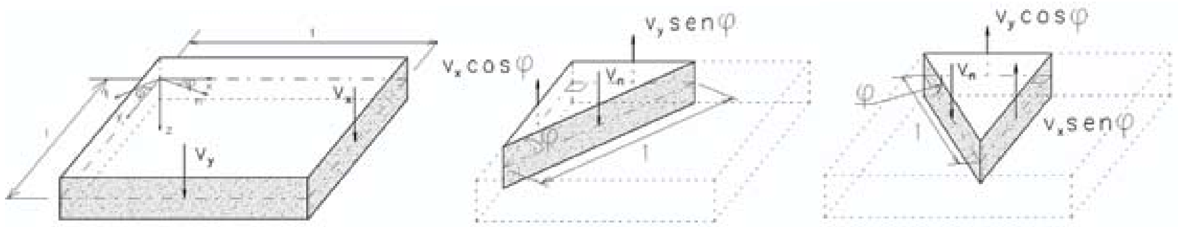


Figure 11 – (a) transverse shear force components in the inner layer of a shell element; free-body diagram for determination of (b) v_n and (c) v_t



(a) (b) (c)

4.3 Dimensioning the Inner Layer

The inner layer must transmit transverse shear forces. In the following, we shall analyze the behavior of an inner layer with unit length along the *x*- and *y*-axes, thickness t_c, subjected to shear forces orthogonal to the element's plane (v_x and v_y), as in Fig. 11. The thickness of the inner layer is t_c, but it is accepted that the shear forces act over a lever arm z_c..
Let us consider an arbitrary rotation of the orthogonal *x*- and *y*-axes about the *z* axis, characterized by an angle φ, and the axes *n* and *t,* mutually orthogonal, then defined. The equilibrium of the vertical forces acting on the elements of unit length in the *n*- (figure 11b) e *t*-direction (figure 11c) gives the following transformation equations for the transverse shear forces components:

$$v_n = v_x \cos\varphi + v_y sen\varphi \; ; \qquad v_t = -v_x sen\varphi + v_y \cos\varphi$$

The sum of the squares of the anterior equations does not depend on φ:

$$v_n^2 + v_t^2 = v_x^2 + v_y^2 = v_o^2 \tag{4.5}$$

In particular, when $v_n=v_o$, we get $\varphi=\varphi_0$ and $v_m=0$. In this case, we may define the principal shear direction, identified by the angle φ_0 and determined by:

$$tg\varphi_0 = \frac{v_y}{v_x} \tag{4.6}$$

The principal shear force and its direction can, therefore, be determined from the v_x and v_y values. In the following, we distinguish two possible mechanisms for the shear forces transfer.
a) <u>case in which no specific shear reinforcement is required</u>
When dimensioning slab, we usually limit the nominal shear stress acting on critical defined sections as to omit shear reinforcement steel. In this case, it must be verified that:

$$v_0 \leq \frac{V_{Rd1}}{b} \tag{4.7}$$

CEB (1990) specifies the use of the following expression for the calculation of the shear resistance V_{Rd1} for members with parallel chords and concrete limited to 50 MPa:

$$V_{Rd1} = 0{,}12\xi(100\rho f_{ck})^{1/3} b_{red} d \quad \textbf{(4.8)}$$

where $\xi = 1+\sqrt{200/d}$; d is the effective depth of the element, in mm; b_{red} is the reduced width of the section equal to the full breadth minus the sum of the widths of the widths of tendon ducts situated within the section; ρ is the ratio of bonded flexural tensile reinforcement in the principal direction extending for a distance at least equal to d beyond the section considered, except at end supports where the extension may be considered adequate if the length of bar beyond the centre-line of support is equal to at least 12 times the diameter. For an element fitted with reinforcement running along two orthogonal directions x and y and geometric ratio ρ_x and ρ_y, as indicated in figure 4.10, the reinforcement ratio in the principal direction shall be calculated as follows:

$$\rho = \rho_x \cos^2\varphi + \rho_y sen^2\varphi \, ; \quad \rho_x = \frac{A_{sx}}{b_x d} ; \rho_y = \frac{A_{sy}}{b_y d} \quad \textbf{(4.9)}$$

b) <u>case in which specific shear reinforcement must be provided</u>

When the equation (4.7) cannot be applied, the resisting mechanism should be analogous to that of a beam, locally oriented according to the principal shear direction. The diagonal compression field, $v_0/sen\vartheta$, makes an angle ϑ with the xy plan, and is the resultant of the sum of two component forces: $v_0 cotg\vartheta$ parallel to the xy plane, and v_0, parallel to the tz plane. Assuming that vertical stirrups are used, the following equations must be verified:

• diagonal compressive forces in concrete:

$$F_{Scw} = \frac{v_0}{sen\vartheta} \leq F_{Rcw} = f_{cd2} z_c \cos\vartheta \quad \textbf{(4.10)}$$

• tensile forces in the web reinforcement (stirrups):

$$F_{Stw} = v_0 \leq F_{Rtw} = \frac{A_{sw} f_{ywd}}{s} z_c \cot g\vartheta \quad \textbf{(4.11)}$$

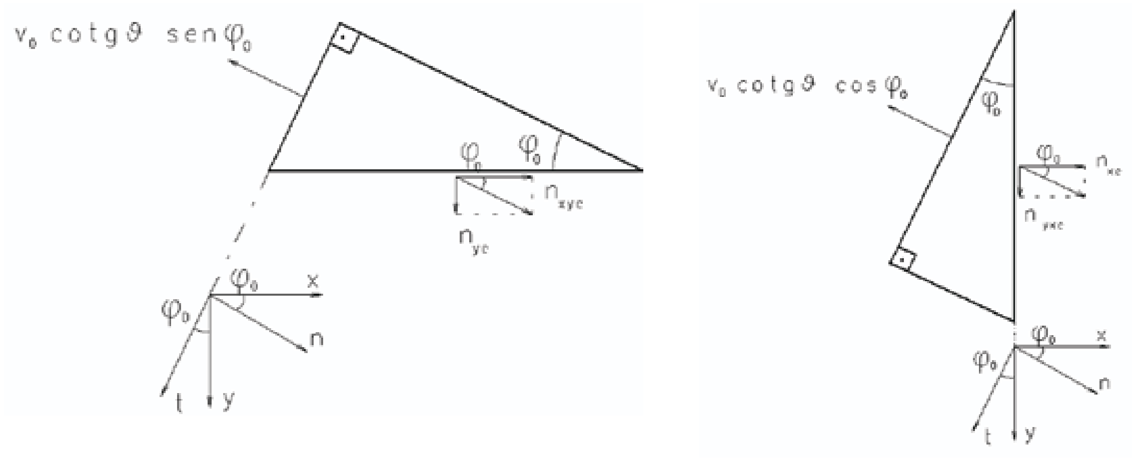
• additional truss axial force in the tension and compression chords (outer layer of the model):

$$\Delta F = \Delta F_s + \Delta F_i = v_o . \cot g\vartheta \quad \textbf{(4.12)}$$

Angle ϑ is subjected to the same limitations as applied to linear elements subjected to shear forces. According to NBR-6118 [4], particularly, it can be chosen freely within the limits $30^\circ \leq \vartheta \leq 45^\circ$. The selection of ϑ must be based primarily on practical considerations on detailing. A low value of ϑ allows for large stirrup spacing and facilitates casting of concrete, but require more longitudinal reinforcement.

The vertical truss force $v_0 . cotg\vartheta$ (equation 4.12), acting on an element of unit length in the n-direction in the inner layer, must be in equilibrium with forces acting on the outer layers of the three-layer model. The components of these forces are calculated for elements with unit length parallel to the x- and y- directions so that they can then be distributed be-

Figure 12 – Equilibrium of principal shear components forces in a membrane element with unit length (a) in the x-direction; (b) in the y-direction



(a) (b)

tween the upper and lower layers. By increasing the acting forces on the upper and lower layers, one can guarantee the adequate anchorage of the reinforcement steel.

Let us consider, first of all, a prism obtained from a shell element delimited by a plane parallel to *x*-axis and two vertical planes: one orthogonal and the other parallel to the principal shear direction (figure 12a). Imposing the conditions of equilibrium in the element, of unit length along *x*, we can determine n_{yc} and n_{xy}, which are the acting forces in the *x*-direction per unit length.

$$n_{yc} = v_o \cot g\vartheta . sen^2\varphi_o = v_o \cot g\vartheta . \frac{tg^2\varphi_o}{1+tg^2\varphi_o} = v_o \cot g\vartheta . \frac{v_y^{\,2}}{v_x^{\,2}+v_y^{\,2}} = \frac{v_y^{\,2}}{v_o}\cot g\vartheta$$

$$n_{xyc} = v_o \cot g\vartheta . sen\varphi_o . \cos\varphi_o = v_o \cot g\vartheta . \frac{tg\varphi_o}{1+tg^2\varphi_o} = v_o \cot g\vartheta . \frac{v_x v_y}{v_x^{\,2}+v_y^{\,2}} = \frac{v_x v_y}{v_o}\cot g\vartheta$$

Let us now consider a prism obtained from a shell element delimited by a plan parallel to axis the *y*-direction and two other planes: one orthogonal and the other parallel to the principal shear direction. Proceeding in a similar way as in the previous case, we obtain:

$$n_{xc} = \frac{v_x^{\,2}}{v_o}\cot g\vartheta; \qquad n_{yxc} = n_{xyc} = \frac{v_x v_y}{v_o}\cot g\vartheta$$

It is important to notice that n_{xc}, n_{yc} and n_{xyc} represent the global contributions due to the truss mechanism, and must be still divided between the upper and lower layer.

4.4 Dimensioning the thickness of the different layers

For the sake of simplicity, we refer to the middle plane of the reinforcements in the *x* and *y*-directions and assume the thickness of the outer layers to be twice the distance between the middle plane of the reinforcement in the *x*- and *y*-directions and the external side of the element.

Consequently, the distinction between $y_{x,s}$ and $y_{y,s}$ and between $y_{x,i}$ and $y_{y,i}$ no longer apply:

$$y_{xs} = y_{ys} = y_{ns}; \quad y_{xi} = y_{yi} = y_{ni}; \quad y_{xys} = y_{yxs} = y_{ts}; \quad y_{xyi} = y_{yxi} = y_{ti}$$

$$z_x = z_y = z_n = y_{ns} + y_{ni}; \quad z_{xy} = z_t = y_{ts} + y_{ti}$$

and the internal forces acting on the outer layers are:
• when no shear reinforcement is required due to the effect of v_x and v_y:

$$n_{Sdx,s} = n_{Sdx}\frac{z-y_s}{z} - \frac{m_{Sdx}}{z}\;;\quad n_{Sdx,i} = n_{Sdx}\frac{z-y_i}{z} + \frac{m_{Sdx}}{z}\;;\quad n_{Sdy,s} = n_{Sdy}\frac{z-y_s}{z} - \frac{m_{Sdy}}{z}$$

$$n_{Sdy,i} = n_{Sdy}\frac{z-y_i}{z} + \frac{m_{Sdy}}{z}\;;\quad v_{Sd,s} = v_{Sd}\frac{z-y_s}{z} - \frac{m_{Sdxy}}{z}\;;\quad v_{Sd,i} = v_{Sd}\frac{z-y_i}{z} + \frac{m_{Sdxy}}{z}$$

• when shear reinforcement is required due to the effect of v_x and v_y:

$$n_{Sdx,s} = n_{Sdx}\frac{z-y_s}{z} - \frac{m_{Sdx}}{z} + \frac{v_x^2}{v_o}\cot g\vartheta\frac{z-y_s}{z}\;;\quad n_{Sdx,i} = n_{Sdx}\frac{z-y_i}{z} + \frac{m_{Sdx}}{z} + \frac{v_x^2}{v_o}\cot g\vartheta\frac{z-y_i}{z} \tag{4.13}$$

$$n_{Sdy,s} = n_{Sdy}\frac{z-y_s}{z} - \frac{m_{Sdy}}{z} + \frac{v_y^2}{v_o}\cot g\vartheta\frac{z-y_s}{z}\;;\quad n_{Sdy,i} = n_{Sdy}\frac{z-y_i}{z} + \frac{m_{Sdy}}{z} + \frac{v_y^2}{v_o}\cot g\vartheta\frac{z-y_i}{z} \tag{4.14}$$

$$v_{Sd,s} = v_{Sd}\frac{z-y_s}{z} - \frac{m_{Sdxy}}{z} + \frac{v_x v_y}{v_o}\cot g\vartheta\frac{z-y_s}{z}\;;\quad v_{Sd,i} = v_{Sd}\frac{z-y_i}{z} + \frac{m_{Sdxy}}{z} + \frac{v_x v_y}{v_o}\cot g\vartheta\frac{z-y_i}{z} \tag{4.15}$$

If concrete strength requirement is not satisfied when operating in this manner on account of the reduced thickness taken on by the outer layers, it is possible to adopt the following procedure:
• increase the concrete cover, accepting a reduction of the internal lever arm and a consequent increase in the reinforcement;
• increase the layer thickness and leave unchanged the position of the reinforcement which, therefore, becomes eccentric relative to the layer. This means that the amount of reinforcement provided has to be changed so as to restore equilibrium conditions. This variation can be assessed with the aid of the mechanism described below, which concerns the entire three layer model (figure 13a). Equilibrium must be restored in both reinforcement directions.
For reinforcement positioned in the middle plane of the outer layers (figure 13a), the forces $n_{Rd,s}$ and $n_{Rd,i}$ resisted by the rebars produce the following moment referring to point P:

$$M_P = n_{Rd,s}\left(h - \frac{t_s}{2} - b_i^{'}\right) + n_{Rd,i}\left(\frac{t_i}{2} - b_i^{'}\right)$$

Assuming that the reinforcement will be positioned at a distance of b'_s and b'_i from the upper and lower edges respectively, the new forces resisted by the reinforcement produce the following moment, also referring to point P:

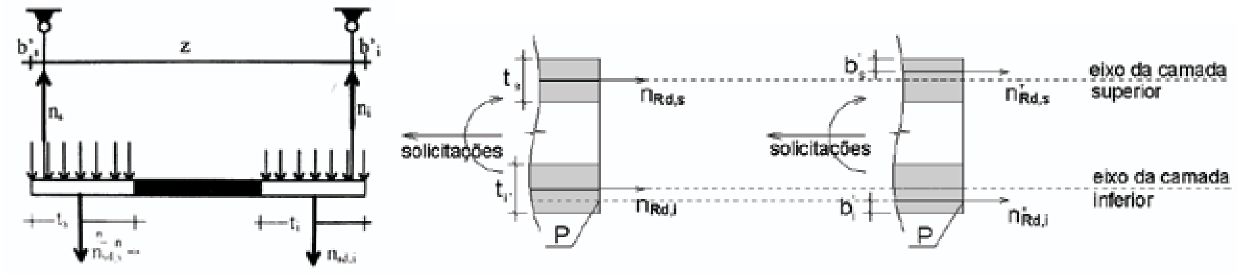
$$M_P = n_{Rd,s}^{*}\left(h - b_s^{'} - b_i^{'}\right)$$

Yielding the two previous expressions, we determine the new forces acting on the reinforcements:

$$n_{Rd,s}^{*} = \frac{n_{Rd,s}\left(h - \frac{t_s}{2} - b_i^{'}\right) + n_{Rd,i}\left(\frac{t_i}{2} - b_i^{'}\right)}{h - b_s^{'} - b_i^{'}}\;;\quad n_{Rd,i}^{*} = n_{Rd,s} + n_{Rd,i} - n_{Rd,s}^{*} \tag{4.16}$$

The intermediate layer must be checked for an additional off-plane transverse shear force corresponding to the force transferred between the two reinforcement levels. Besides, the shear force in the inner layer must be reviewed, as for the z_c value.

Figure 13 – Shell element equilibrium: (a) reinforcement in the middle plane of the outer layers; (b) reinforcement eccentric relative to the outer layers



(a) (b)

4.5 Automatic design

Lourenço & Figueiras [2] developed an important design tool by proposing an automatic procedure for the design of the different cases, according to the need of reinforcement in each of the outer layers, of a shell element in a computer program. The sub-routine "Shell.bas", along with another sub-routine for the design of membrane elements ("Membrane.bas"), developed by Lourenço & Figueiras was adapted by the authors in Visual Basic code and implemented in a Microsoft Excel worksheet for evaluation and tests. Different from the simplified manual procedure, the compressive and tensile resultants on a layer may not to be, necessarily, on the same level, but the global equilibrium of the element is guaranteed in all iteration.

5 Design of plate elements

The main methods for the design of plate elements are the Wood method [19], which is based on the normal moment verification, and the method based on the forces equilibrium, proposed by Brondum-Nielsen [13], developed and adopted by the CEB [1]. In a work by Parsekian [21], the Wood method is studied in depth, and an evaluation of the torsional moments in the design of reinforced concrete is made.

5.1 Equations of Wood's Method

Amongst several methods for the design of slab elements of reinforced concrete with orthogonal reinforcement net that take into account both flexural and torsional moments deriving from an elastic structural analysis, the most diffused is Wood's [19]. His design method, which is based on the normal moment yield criterion, provides good results for lightly reinforced elements.
This method consists of determining the minimum reinforcement quantity so that in one point of the slab any normal component of the ultimate resistant moment is always greater than the normal solicitant component. The ultimate normal resistant moment m_{Rn}, acting in the n-direction along the yield line, is found by equilibrium conditions in one infinitesimal element in the n-axis direction, given the ultimate resistant moments in the directions of the reinforcements per unit length:

$$m_{Rn} = m_{Rx} \cos^2\theta + m_{Ry} sen^2\theta \qquad (5.1)$$

The solicitant moments transform as follows for the n- direction (equilibrium condition):

$$m_{Sn} = m_{Sx} \cos^2\theta + m_{Sy} sen^2\theta - m_{Sxy} sen 2\theta \qquad (5.2)$$

The yield condition is established when the normal solicitant moment equals the ultimate resistant moment, i.e.:

$$m_{Sn} = m_{Rn}$$

The equations of this method are obtained by the minimization of both the difference between the normal component of the ultimate resistant moment provided by the reinforcement and the normal component of the moment given by the stress field and the total reinforcement amount required. Expressions for the determination of the ultimate resistant moments in the rebar directions are proposed as for the design for flexure. Membrane effects in the slab and interaction between forces acting on the rebars arranged on opposite sides of the slab are not considered.

Positive moment fields

In this case, it is necessary that $m_{Rn} - m_{Sn} \geq 0$ and the resistant moments for the dimensioning of positive reinforcements are:

$$m_{Rx,i} = m_{Sx} + \left|m_{Sxy}\right|; \quad m_{Ry,i} = m_{Sy} + \left|m_{Sxy}\right|$$

Negative moment fields

In this case, we must have $m_{Rn} \leq m_{Sn}$ and the most economic arrangement for the reinforcement is obtained when:

$$m_{Rx,s} = m_{Sx} - \left|m_{Sxy}\right|; \quad m_{Ry,s} = m_{Sy} - \left|m_{Sxy}\right| \tag{5.3}$$

Mixed (positive and negative) moment fields

When applying the expressions above, it is expected that the design moments be positive and negative, respectively. However, in some cases, these equations lead to design moments with opposite signs, i.e., we may encounter negative moments for positive reinforcement and vice-versa. This occurs due to the fact that the yield criterion defined by Johansen does not include the case of ultimate resistant moments with opposite signs. In these cases, the following procedure should be followed:

Correction for the inferior side of the slab, associated to the positive reinforcement

i) if $m_{Rx,i} < 0$, adopt $m_{Rx,i} = 0$, and we get: $m_{Ry,i} = m_{Sy} + \left|\dfrac{m_{Sxy}^{2}}{m_{Sx}}\right|$

If the value of $m_{Ry,i}$ in this equation results negative, no reinforcement is required.

ii) if $m_{Ry,i} < 0$, adopt $m_{Ry,i} = 0$, and we get: $m_{Rx,i} = m_{Sx} + \left|\dfrac{m_{Sxy}^{2}}{m_{Sy}}\right|$

If the value of $m_{Rx,i}$ in this equation results negative, no reinforcement is required.

Correction for the upper side of the slab, associated to the negative reinforcement

i) if $m_{Rx,s} > 0$, adopt $m_{Rx,s} = 0$, and we get: $m_{Ry,s} = m_{Sy} - \left|\dfrac{m_{Sxy}^{2}}{m_{Sx}}\right|$

If the value of $m_{Ry,s}$ in this equation results positive, no reinforcement is required.

ii) if $m_{Ry,s} > 0$, adopt $m_{Ry,s} = 0$, and we get: $m_{Rx,s} = m_{Sx} - \left|\dfrac{m_{Sxy}^{2}}{m_{Sx}}\right|$

If the value of $m_{Rx,s}$ in this equation results positive, no reinforcement is required.

5.2 Three-layer model

The equations proposed by Wood leads to less conservative results for increasing reinforcement ratio and for increasing angles between the principal moment and the reinforcement directions. Gupta demonstrated that the expressions of Wood are approximated because they do not consider the influence of the different lever arms of the internal forces. Marti [17] demonstrated that the criterion of the normal yield moment overestimate the resistance of slab elements that are subjected to significant torsional moments. From the exposed above, it is made evident the necessity of an alternative model for the design of plate elements.
The three-layer model proposed by the CEB [1] for the design of slab elements was developed as an alternative to Wood's approach. It is a particular case of the three-layer model for the design of shell elements. The forces acting on the upper and lower layers of the model are given by the expressions:

$$n_{Sdx,s} = -\frac{m_{Sdx}}{z_x} + \frac{v_x^2}{v_o}\cot g\vartheta\,\frac{z_x - y_{x,s}}{z_x}\,; \qquad n_{Sdx,i} = \frac{m_{Sdx}}{z_x} + \frac{v_x^2}{v_o}\cot g\vartheta\,\frac{z_x - y_{x,i}}{z_x} \tag{5.4}$$

$$n_{Sdy,s} = -\frac{m_{Sdy}}{z_y} + \frac{v_y^2}{v_o}\cot g\vartheta\,\frac{z_y - y_{y,s}}{z_y}\,; \qquad n_{Sdy,i} = \frac{m_{Sdy}}{z_y} + \frac{v_y^2}{v_o}\cot g\vartheta\,\frac{z_y - y_{y,i}}{z_y} \tag{5.5}$$

$$v_{Sd,s} = -\frac{m_{Sdxy}}{z_{xy}} + \frac{v_x v_y}{v_o}\cot g\vartheta\,\frac{z_{xy} - y_{xy,s}}{z_{xy}}\,; \qquad v_{Sd,i} = \frac{m_{Sdxy}}{z_{xy}} + \frac{v_x v_y}{v_o}\cot g\vartheta\,\frac{z_{xy} - y_{xy,i}}{z_{xy}} \tag{5.6}$$

The outer layers must be verified as membranes subjected to in-plane actions, as forces per unit length, n_{Sdx}, n_{Sdy} and v_{Sd}.

6 Numerical examples

6.1 Design of a membrane element

Design of a membrane element with orthogonal mesh of rebars (*m* e *n* directions). The solicitant stresses are principal stresses: n_{Sdx}=219,9 kN/m and n_{Sdy}=119,1 kN/m. The thickness of the element is 0,10m and f_{ck}=20 MPa. Several design cases were studied. In each case the reinforcement mesh deviate from the principal direction (the reinforcement in the *m* direction makes an angle α against the x-axis). The total reinforcement required are plotted as a function of x in figure 14.

6.2 Crack width evaluation – membrane element

A membrane element with thickness=0,10m and concrete with f_{ck}=20MPa is subjected to normal stresses n_{Sdx}=200 kN/m, n_{Sdy}=150 kN/m and shear stresses that vary from 150 to 400 kN/m. The reinforcement is determined and the crack widths are calculated by the three proposed methods, assuming that φ10mm rebars are used. In figure 15 one can find the crack widhts as a function of shear stresses v_{Sd}.

6.3 Design of plate elements

Case I: plate element, thickness=0,15m, f_{ck}=25MPa, subjected to m_{Sdx}=50 kN.m/m, m_{Sdy}=45 kN.m/m and m_{Sdxy} varying from 13 to 28 kN.m/m (positive bending moments in both directions).

Case II: plate element, thickness=0,15m, f_{ck}=25MPa, subjected to m_{Sdx}=25 kN.m/m, m_{Sdy}=-31 kN.m/m and m_{Sdxy} varying from -8 to -23 kN.m/m (mixed positive and negative bending moment field).
It is assumed, in both cases, d'=0,03m. The element was designed by Wood´s method and through the use of Lourenço´s routine. Total reinforcement amount required are shown as a function of the torsional moment in figure 16.

Figure 14 – Total reinforcement required for a membrane element with orthogonal set of rebars placed in different inclination (α)

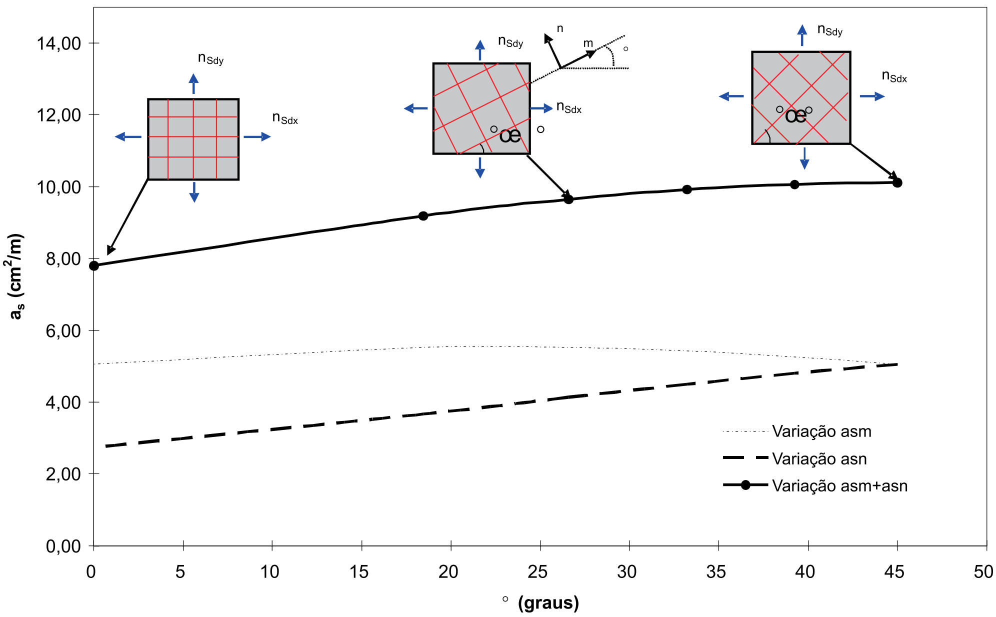


Figure 15 – Crack width in membrane elements according to Gupta, Eurocode and NBR-6118

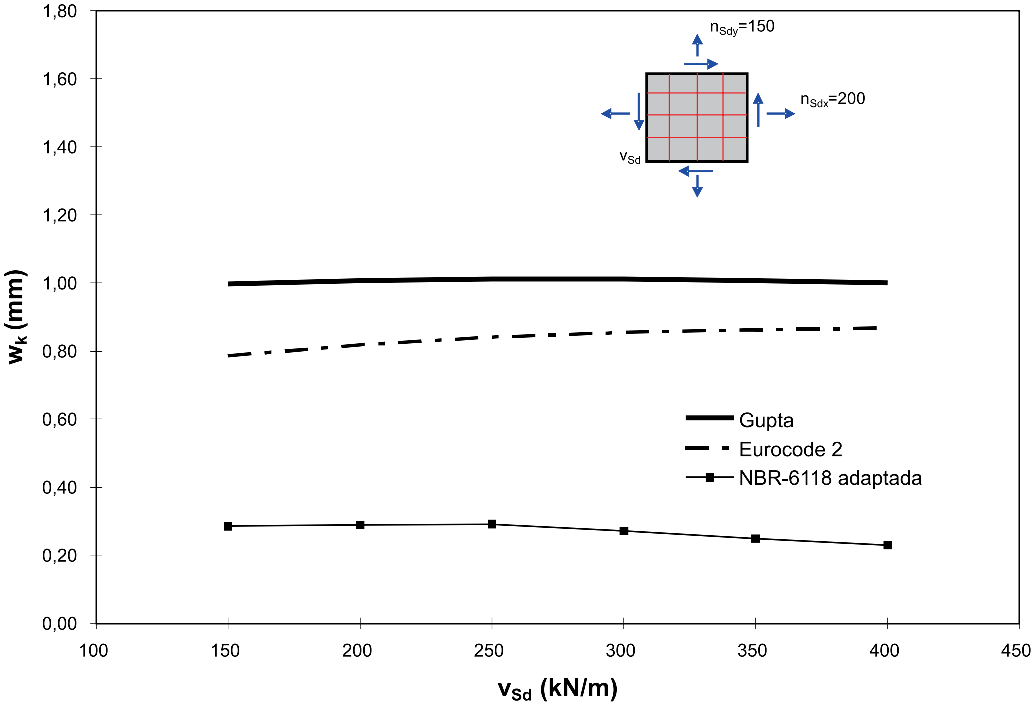
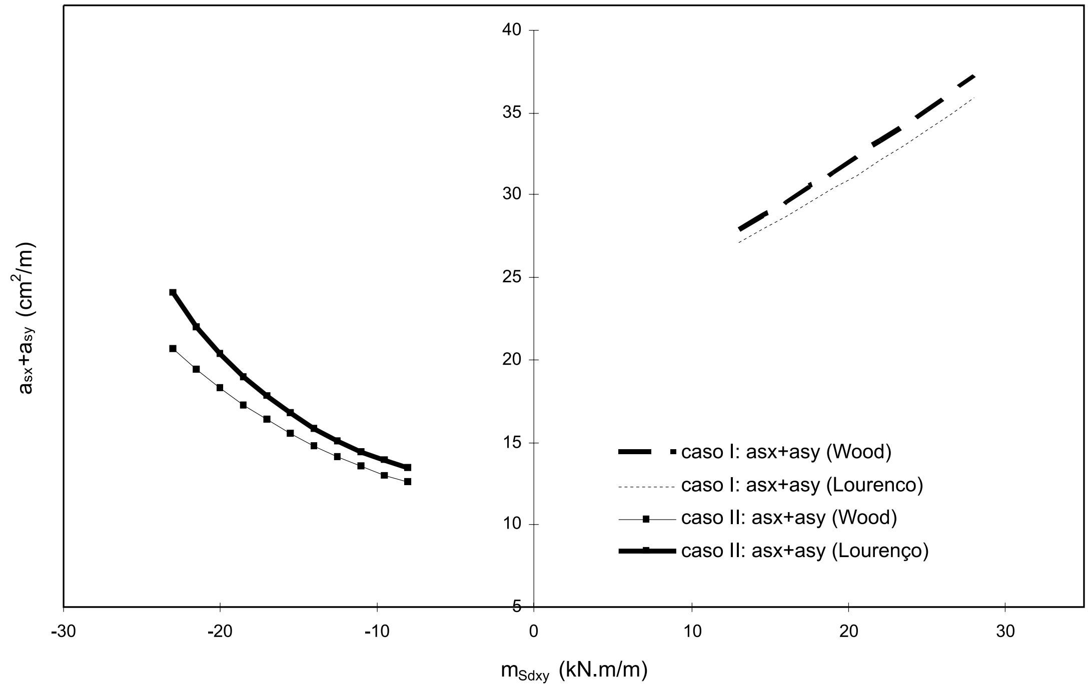


Figure 16 – Total reinforcement requiered in two dimensioning cases: case I – positive bending moments; case II – mixed positive and negative bending moments



7 Results and discussions

Example 6.1 confirms an expected result. The economic design of membrane elements happens when the reinforcement direction coincides with the principal direction. The total reinforcement required increases with higher values of α. Expressions for the calculation of crack widths were applied in example 6.2. For the six cases studied, crack width calculations considering the tension stiffening effect reduced, in average, 17% the expected value if no concrete contribution were taken in account. Crack widths calculations from NBR-6118 adapted expressions lead to even less conservative results. In example 6.3, case I, it is observed that the plate element designed by Lourenço´s routine resulted in lower reinforcement quantities (17%, in average), being more economic. This occurred because the routine identified biaxial compression on the upper layer, and considered an increased value for the compression strength, according to 2.3. In case II, the reinforcement required by Lourenço´s routine increased for higher values of torsional moments. The difference may be due to the adoption of f_{cd2} and due to the more precise consideration of the different lever arm of the internal forces in Lourenço´s design.

8 Conclusions

The following conclusions may be drawn:

- the formulation for the design of membrane elements based on the criterion of Baumann provide solutions that take into account equilibrium and compatibility conditions, and are adequate for ULS verifications;
- the NBR-6118 [4] formulation to estimate crack width is less conservative than the one suggested by the Eurocode 2 [9] ;
- the manual iterative dimensioning is highly simplified if we assume equal values of *y* for orthogonal reinforcements in the same layer and identical lever arms for normal, shear and transversal shear($z=z_c=z_n=z_t$) forces. Proceeding in this manner, however, one shall check for the necessity to establish, in particular cases, equilibrium condition for the eccentric reinforcement. Generally, a final solution is obtained in little iteration;
- the automatic design proposed by Lourenço in an adequate tool for the design of concrete structures. The solutions are more economic, since the thicknesses of the outer layers are always determined for limit compressive actions and, as a consequence, the largest lever arms for internal forces are considered. However, additional verifications must be made for the transverse shear and for the extension of the steel reinforcement (anchorage);

- the design of slab elements by the method of Wood [9], as well as by the simplified method for slabs, provide adequate resistance to lightly reinforced elements. The equations of this method lead to non-conservative results for elements with increasing reinforcement ratios or increasing angle between the reinforcement direction and the principal moment direction since the actual positions of the reinforcement and, consequently, the actual lever arms, are not considered.

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